Qualifying Examination Spring 2009
Algorithms

Do all problems. No books or notes may be used. Your grade depends on the correctness, completeness and style of your answer.

Problem 1

a) (10 points): Fill in the rest of the following table, where each entry should be O, Ω, or Θ, according to whether the row function is O, Ω, or Θ of the column function. If more than one is true, you should put the strongest result possible. All logarithms are base 2.

	$\binom{2n}{2}$	$\binom{2n}{n}$	$2^{\log^2 n}$
$2^{n \log n}$	Ω		
$3^{\log n^2}$			

b) (15 points): Briefly justify each of the five entries that you made in the table for part a). For example, the justification for the pre-entered Ω entry could be something like:

$$\text{“ } 2^{n \log n} = \Omega(\tbinom{2n}{2}), \text{ since } 2^{n \log n} = n^n,\ \tbinom{2n}{2} = 2n^2 - n, \text{ and } n^n > 2n^2 - n \text{ for all } n \geq 3.\text{”}$$

Problem 2

a) (5 points): Define the notion of **strongly connected components** of a directed graph.

b) (10 points): Describe an algorithm to find the strongly connected components of a directed graph $G = (V, E)$ in time linear in the size $|V| + |E|$ of the graph.

c) (20 points): A **2CNF** formula over a set of variables U is a boolean expression consisting of the conjunction of clauses, where each clause is the disjunction of *exactly two literals*, and each literal is either a variable from U or its negation.

Describe an algorithm that decides whether a given 2CNF formula is satisfiable or not in time linear in the number of clauses.

Problem 3 Define the HOMOPOLAR SATISFIABILITY (**HomSAT**) problem as follows:

Instance: A CNF formula over a set of variables U, where each clause is *homopolar*, *i.e.*, it consists of literals that are either all positive or all negative.

Question: Is the formula satisfiable, *i.e.*, does there exist a truth assignment for the variable in U that renders the formula true?

a) (10 points): Define the complexity class **NP**, and show that **HomSAT** is in NP.

b) (25 points): Define the notion of NP-hardness, and show that **HomSAT** is NP-hard.

c) (5 point): Define the notion of NP-completeness, and conclude that **HomSAT** is NP-Complete.