A proof of Ademek's theorem

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1 Introduction

In [AMV07], Adamek et. al. presented a proof that the category of iteration theories is monadic over the category of signatures. They used Beck's theorem [MacL] but needlessly got involved in verifying a long list of identities. We present a short argument.

2 Definitions

We presume familiarity with Lawvere theories and iteration theories [BlEs93].

The category IT of iteration theories has as objects the iteration theories, and as morphisms $\varphi : I \to I'$ the dagger-preserving theory morphisms.

A **signature** is a collection $\Sigma = (\Sigma_{n,p})$, of pairwise disjoint sets $\Sigma_{n,p}$, $n, p \geq 0$. A **signature morphism** $h : \Sigma \to \Sigma'$ is a collection of functions $f_{n,p} : \Sigma_{n,p} \to \Sigma'_{n,p}$, $n, p \geq 0$.

Thus signatures and signature morphisms form a category, Sig.

Let $U : \mathsf{IT} \to \mathsf{Sig}$ be the functor which maps the iteration theory T to the signature $T(n,p)_{n,p}$, and an iteration theory morphism $\varphi : T \to T'$, the collection of functions $\varphi : T(n,p) \to T'(n,p)$.

Theorem 2.1 (Beck) *Suppose $U : A \to B$ is a functor, with a left adjoint*

F. Let $T = B \xrightarrow{F} A \xrightarrow{U} B$. Then U is monadic, i.e., A is isomorphic to the category of T-algebras, iff U creates coequalizers for parallel pairs fU, gU having a split coequalizer in B.

We start with a warm-up.

Theorem 2.2 *The functor $U : Th \to \mathsf{Sig}$ has a left adjoint and is monadic.*

Proof. We use Beck's theorem. Suppose $f, g : T \to T'$ are theory morphisms and $e : T'U \to S, s : S \to T'u, t : T'U \to Tu$ are morphisms in Sig such that

$$\begin{aligned} fU \cdot e &= gU \cdot e \\ s \cdot e &= \mathbf{id}_S \\ t \cdot fU &= e \cdot s \\ t \cdot gU &= \mathbf{id}_{T'U}. \end{aligned}$$

We show S may be made into a theory a theory in such a way that e becomes a theory morphism, i.e., $e = e'U$, for some $e' : T' \to S$, and e' is a coequalizer of f, g in TH.

For composable α, β in S, **define**

$$\alpha \circ \beta \quad := \quad (\alpha s \cdot \beta s)e.$$

Now, we show e preserves composition. Indeed, if $\alpha : n \to p$ and $\beta : p \to q$ in T',

$$\begin{aligned} \alpha e \circ \beta e &= ((\alpha e s) \cdot (\beta e s))e \\ &= ((\alpha t f U) \cdot (\beta t f U))e, \quad [e \cdot s = t \cdot fU] \\ &= ((\alpha t \cdot \beta t) f U)e, \quad [f \text{ is morphism}] \\ &= ((\alpha t \cdot \beta t) g U)e, \quad [fU \cdot e = gU \cdot e] \\ &= ((\alpha t g U) \cdot (\beta t g U))e, \quad [g \text{ is morphism}] \\ &= (\alpha \cdot \beta)e, \quad [t \cdot gU = \mathbf{id}]. \quad \Box \end{aligned}$$

This is the ONLY possible definition of $\alpha \circ \beta$, since if $\circ$ is any operation such that e preserves composition,

$$\begin{aligned} \alpha \circ \beta &= \alpha s e \circ \beta s e, \quad [\mathbf{id} = s \cdot e] \\ &= (\alpha s \cdot \beta s)e, \end{aligned}$$

since e preserves composition.

Define the base morphisms in S as αe, for α base in T'. The rest follows, since e is surjective, and thus preserves equations.

Now we show e is a coequalizer of f, g in TH. Suppose $f \cdot \overline{e} = g \cdot \overline{e}$ in TH. Then

$$\begin{array}{rcl} t \cdot fU \cdot \overline{e}U & = & t \cdot gU \cdot \overline{e}U \\ \overline{e}U & = & (e \cdot s) \cdot \overline{e}U \\ & = & e \cdot (s \cdot \overline{e}U). \end{array}$$

Now we show $r = s \cdot \overline{e}U$ is a morphism in TH.

$$\begin{array}{rcl} \alpha r \cdot \beta r & = & (\alpha s \overline{e}U) \cdot (\beta s \overline{e}U), \quad [r = s \cdot \overline{e}U] \\ & = & (\alpha s \cdot \beta s)\overline{e}U, \quad [\overline{e} \text{ is morphism in } \mathsf{TH}] \\ & = & (\alpha s \cdot \beta s)er, \quad [\overline{e}U = e \cdot r] \\ & = & (\alpha se \cdot \beta se)r, [e \text{ is morphism in } \mathsf{TH}] \\ & = & (\alpha \cdot \beta)r, \quad [s \cdot e = \mathbf{id}]. \quad \square \end{array}$$

Theorem 2.3 *$U : \mathsf{IT} \to \mathsf{Sig}$ has a left adjoint, the functor R, which maps a signature Σ to the iteration theory of rational $\Sigma_\perp$-trees.*

Theorem 2.4 *[AMV07] U is monadic.*

Proof. Again we use Beck's theorem, and show that if $f, g : T \to T'$ are iteration theory morphisms, and e, s, t are as above, then there is a unique way to define † on S such that e preserves †, and e becomes an iteration theory morphism.

For $\alpha : n \to n + p$ in S, define

$$\alpha^\dagger \quad := \quad ((\alpha s)^\dagger)e.$$

We now show e preserves †. Suppose $a : n \to n + p$ in T'.

$$\begin{array}{rcl} (ae)^\dagger & = & (((ae)s)^\dagger)e \\ & = & ((atfU)^\dagger)e, \quad [e \cdot s = t \cdot fU] \\ & = & ((at)^\dagger fU)e, \quad [f \text{ is morphism in } \mathsf{IT}] \\ & = & ((at)^\dagger gU)e, \quad [fU \cdot e = gU \cdot e] \\ & = & ((atgU)^\dagger)e, \quad [g \text{ is morphism in } \mathsf{IT}] \\ & = & (a^\dagger)e, \quad [t \cdot gU = \mathbf{id}]. \quad \square \end{array}$$

Again, we must show e is a coequalizer of f, g in IT, but the argument is the same as above. $\square$

References

[AMV07] J. Adámek, S. Milius, and J. Velebil. What are iteration theories? Manuscript, May, 2007.

[BlEs93] S.L. Bloom and Z. Ésik. *Iteration Theories.* Springer, 1993.

[MacL] S. Mac Lane. Categories for the Working Mathmatician. Graduate Texts in Mathematics, Springer–Verlag, 1971.