Algebraic words and ordinals

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Abstract

We consider systems of fixed point equations, such as

$$\begin{aligned} F_1 &= a \cdot F_1 \\ F_2(x) &= x \cdot F_2(F_1 \cdot x) \end{aligned}$$

and their solutions in "categorical algebras", especially trees and words.

- Using equations, we define “algebraic” and “regular” elements of categorical algebras.

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and their solutions in "categorical algebras", especially trees and words.

- Using equations, we define “algebraic” and “regular” elements of categorical algebras.
- Algebraic and regular ordinals will be discussed.

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- is system

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- $\Sigma \cup \{F_1, \ldots, F_n\}$ is ranked alphabet, F_i has rank k_i and
- t_i is **term** built from variables $x_1, \ldots, x_{k_i}$ and function symbols in $\Sigma \cup \{F_1, \ldots, F_n\}$. [back reg]

Examples

- $$F_1 \;=\; a \cdot F_1.$$

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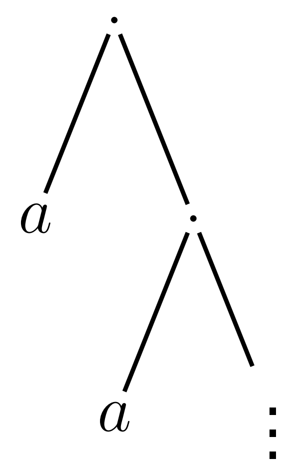

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$$F_1 \quad = \quad a \cdot a \cdot a \cdots$$

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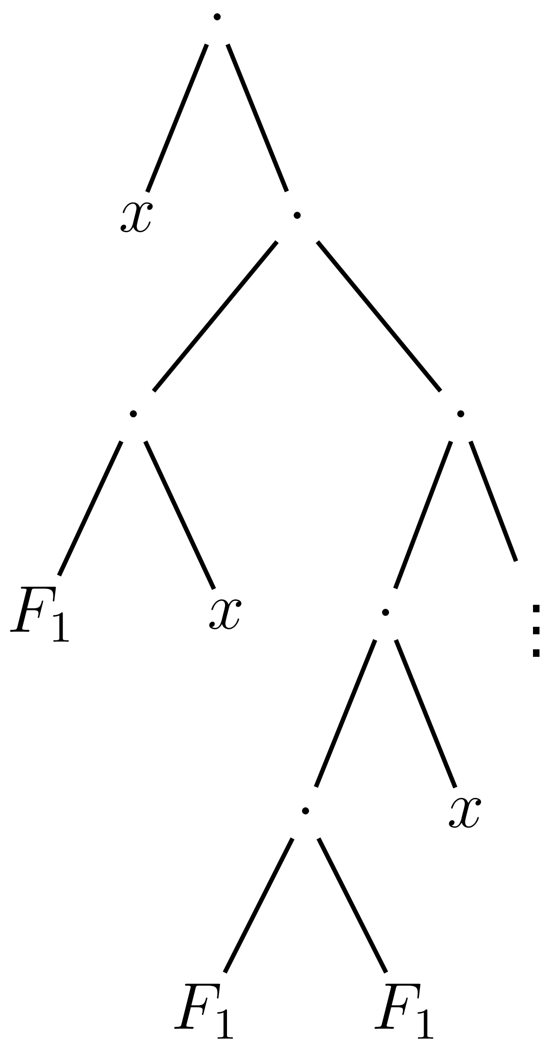
$$\begin{aligned} F_1 &= a \cdot a \cdot \ldots \\ F_2(x) &= x \cdot (F_1 \cdot x) \cdot (F_1^2 \cdot x) \cdot (F_1^3 \cdot x) \ldots \end{aligned}$$

A solution in trees

$$F_2(x) \quad =$$



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$$h^n \circ \sigma^B \;\; = \;\; \sigma^A \circ h,$$

up to natural isomorphism.

Special case

- An ordered Σ-algebra $\mathcal{A}$ consists of a partially ordered set A and order preserving functions $\sigma^A : A^n \to A$, $\sigma \in \Sigma_n$.

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- An ordered Σ-algebra $\mathcal{A}$ consists of a partially ordered set A and order preserving functions $\sigma^A : A^n \to A$, $\sigma \in \Sigma_n$.
- Any ordered Σ-algebra $\mathcal{A}$ is a cΣa, where there is an arrow $a \to b$ iff $a \leq b$.

Continuous cΣa's

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Continuous ordered algebras

An ordered algebra is continuous iff it has a least element, sups of ω-chains, and all functions σ^A preserve sups of ω-chains.

The signature $\Sigma(A)$

For a finite set A, $\Sigma(A) = \{\cdot\} \cup A$ is the signature with

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- a constant symbol a for each $a \in A$.
- Words (and trees) on A determine continuous categorical $\Sigma(A)$-algebra, W_A.

Words on $A = \{a_1, \ldots, a_m\}$

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- $a \in A$, the one point word

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If $\mathcal{A}, \mathcal{B}$ are continuous cΣa's, so are

- $\mathcal{A} \times \mathcal{B}$
- $[\mathcal{A} \to \mathcal{B}]$, continuous functors $\mathcal{A} \to \mathcal{B}$, where, e.g., if σ has rank 2, and $f, g : \mathcal{A} \to \mathcal{B}$,

$$\sigma(f, g)(a) \quad := \quad \sigma_B(fa, ga).$$

Fixed points

- If $F : \mathcal{A} \to \mathcal{A}$ is continuous endofunctor on ccΣa, F has an initial fixed point: the colimit of

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- Each (right side) of a system determines endofunctor

$$E_A : \mathcal{A}^\rho \quad \to \quad \mathcal{A}^\rho,$$

where

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- Thus E_A has initial fixed point.

Example

- The system

$$\begin{aligned} F_1 &= a \cdot F_1 \\ F_2(x) &= x \cdot F_2(F_1 \cdot x) \end{aligned}$$

determines the endofunctor on $\mathcal{A} \times [\mathcal{A} \to \mathcal{A}]$:

$$(u, G) \mapsto (a \cdot u,\ \lambda x(x \cdot G(u \cdot x)))$$

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Examples of regular systems

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  is
- written $[a]^\eta$.

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- up to isomorphism, there is one such word.
- For words $u_1, \ldots, u_m$, $[u_1, \ldots, u_m]^\eta$ is word obtained from $[a_1, \ldots, a_m]^\eta$ by substituting the word u_i for the letter a_i in $[a_1, \ldots, a_m]^\eta$.

shuffle and equations

- $[a]^\eta$ is initial solution to

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  $$u \cdot v, u^{\omega}, u^{\omega^{op}}, [\![u_1, \ldots, u_k]\!]^{\eta},\ k \geq 1$$

  [Heilbrunner]
- w isomorphic to $w(L_{a_1}, \ldots, L_{a_m})$, for some regular sets L_{a_i} (not all $\emptyset$), where

$$w = w(L_{a_1}, \ldots, L_{a_m})$$

Suppose $L_a \subseteq \{0,1\}^*$ for each letter in A.

- sets L_a are pairwise disjoint

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- $u <_\ell v \in \{0,1\}^*$ (lexicographic order) if $u <_p v$, i.e., $v = uw$, $w \neq \epsilon$, (prefix order) or $u <_s v$, i.e., $u = u_1 0 u_2$, $v = u_1 1 v_2$, some u_1, u_2, v_2 (strict order).

Related result

Theorem [BE 2005]

- There is a natural set of axioms Ax such that for two terms s, t built from the alphabet A and regular operations, $s = t$ holds in the ccΣa of words iff $Ax \vdash s = t$.

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- There is a natural set of axioms Ax such that for two terms s, t built from the alphabet A and regular operations, $s = t$ holds in the ccΣa of words iff $Ax \vdash s = t$.
- There is a polynomial time algorithm to decide, given terms s, t, whether

$$s \;=\; t$$

holds in words.

Trees on A

Recall: a tree t is

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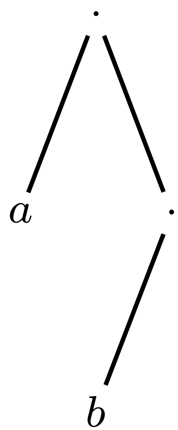
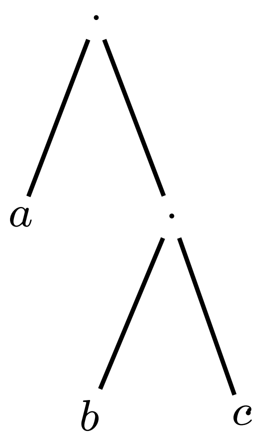
- $t \sqsubseteq t'$ if $t(u) \downarrow \implies t(u) = t'(u)$.
- there is arrow $t \to t'$ if $t \sqsubseteq t'$.

Examples

The domain of $t =$



is $\{\epsilon, 0, 1, 10\}$. (ϵ - empty word). $t \sqsubseteq t'$, where t' is



Trees form (ordered) ccΣa

- For $\Sigma = \Sigma(A) = \{\cdot\} \cup A$, if t_0, t_1 are trees, $t = t_0 \cdot t_1$ is partial function

$$\begin{aligned} t(\epsilon) &= \cdot \\ t(iu) &= t_i(u), \quad i = 0, 1,\ u \in \{0,1\}^*. \end{aligned}$$

  In pictures,

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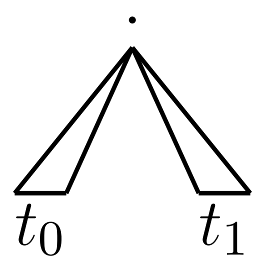

- The tree $a \in A$ is one-point tree

$$a(u) = \begin{cases} a & u = \epsilon \\ \uparrow & \text{otherwise.} \end{cases}$$

yield

The **yield** of a tree t is the word

- $$\text{yield}(t) \quad = \quad (W, <_\ell, \lambda_t)$$

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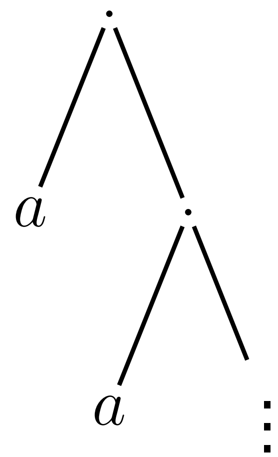

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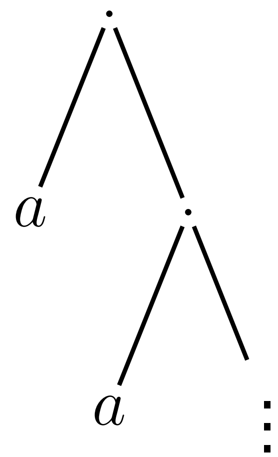

  is

- $$a \cdot a \cdot a \cdots$$

Fact about regular trees

A tree is regular iff it has finitely many subtrees.

Mezei-Wright type theorem

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- E a system of fixed point equations with $k_1 = 0$.

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Then:

- $h(a)$ is isomorphic to b.

Corollary

A word w in W_A is algebraic (regular) iff $w \cong \mathrm{yield}(t)$ for some algebraic (regular) tree t. *[Courcelle]*

Trees and leaf languages

- For a tree t, define $L_a(t) = \{u \in \{0,1\}^* : t(u) = a\}$. Then $\bigcup_a L_a(t)$ is prefix code, and if $a \neq b$, $L_a \cap L_b = \emptyset$.

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- Let L_a, $a \in A$ be a family of pairwise disjoint languages over $\{0,1\}$ such that $\bigcup_a L_a$ is prefix code. Then there is a unique tree t with $L_a(t) = L_a$, each $a \in A$. The domain of t is set of prefixes of $\bigcup_a L_a$.

Algebraic trees and leaf languages

Theorem. [Courcelle, BE] There is an *algebraic* tree t with $L_a(t) = L_a$ for all $a \in A$ iff each L_a is a deterministic cfl.

Ordinals and words

- **Universal property of the lexicographic order:** For every countable linear order $(L, <_L)$ there is a (prefix-free) subset $C \subseteq \{0,1\}^*$ such that

$$(L, <_L) \;\; \cong \;\; (C, <_\ell).$$

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- **Definition**. A countable ordinal α is regular or algebraic if

$$\alpha \;=\; \mathbf{o}(L_w, <_\ell)$$

for some regular or algebraic word w on a one-letter alphabet.

Regular ordinals

- Which ordinals are regular?

Elementary facts

-

$$\begin{aligned} \omega &= \mathbf{o}(1^*0, <_\ell) \\ &= \{0 <_s 10 <_s 110 <_s \ldots\} \end{aligned}$$

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- For X, Y prefix-free subsets of $\{0,1\}^*$, then any word in $X \cdot Y$ has unique decomposition uv, with $u \in X, v \in Y$.

Elementary facts

- $$\begin{aligned} \omega &= \mathbf{o}(1^*0, <_\ell) \\ &= \{0 <_s 10 <_s 110 <_s \ldots\} \end{aligned}$$
- For X, Y prefix-free subsets of $\{0,1\}^*$, then any word in $X \cdot Y$ has unique decomposition uv, with $u \in X, v \in Y$.
- If X, Y are prefix-free subsets of $\{0,1\}^*$ such that $\alpha = \mathbf{o}(X, <_\ell)$ and $\beta = \mathbf{o}(Y, <_\ell)$ are ordinals, then

$$\mathbf{o}(X \cdot Y, <_\ell) = \beta \times \alpha.$$

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$$\mathbf{o}(X \cdot Y, <_\ell) = \beta \times \alpha.$$
- Thus, $\mathbf{o}((1^*0)^n, <_\ell) = \omega^n$, $n \geq 0$.

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Theorem

- An ordinal α is regular iff $\alpha < \omega^\omega$. *[Bloom-Chofrut]*

Algebraic ordinals

Which ordinals are algebraic?

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- Converse? We have only a partial answer.

Ordinal grammars

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- A prefix grammar is an **ordinal grammar** if $\mathcal{L}(S)$ is well-ordered by $<_\ell$.
  Thus, $\mathcal{L}(X)$ is well-ordered by $<_\ell$, for each $X \in N$.

Theorem

An ordinal α is less than $\omega^{\omega^{\omega}}$ iff there is an ordinal grammar G with $\mathbf{o}(\mathcal{L}(G)) = \alpha$.

Open problems

- Is there an algorithm to decide, given two CF-grammars G, G', whether

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- Is every context-free ordinal less than ω^{ω^ω}?
- If $\mathcal{L}(G)$ is a prefix language, is there a prefix grammar G' with

$$(\mathcal{L}(G), <_\ell) \;\cong\; (\mathcal{L}(G'), <_\ell)$$

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