Completing Categorical Algebras

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Abstract

Let Σ be a ranked set. A categorical Σ-algebra, cΣa for short, is a small category C equipped with a functor $\sigma_C : C^n \longrightarrow C$, for each $\sigma \in \Sigma_n$, $n \geq 0$. A continuous categorical Σ-algebra is a cΣa which has an initial object and all colimits of ω-chains, i.e., functors $\mathbb{N} \longrightarrow C$; each functor σ_C preserves colimits of ω-chains. ($\mathbb{N}$ is the linearly ordered set of the nonnegative integers considered as a category as usual.)

It is known that for any category C there is an ω-complete category, unique up to equivalence, which forms a "free ω-completion" of C. We generalize this result to show that any cΣa has a free ω-completion, C^ω, whose algebraic structure is uniquely determined. Further, we generalize the notions of **inequation** and **equation** and show the (in)equations that hold in C also hold in C^ω. We then find examples of this completion when

- C is a cΣa of finite Σ-trees
- C is an ordered Σ algebra

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- C is a cΣa of finite A-synchronization trees
- C is a cΣa of finite words on A.

1 Introduction

Computer science is necessarily concerned with fixed point equations, and in finding settings in which fixed point equations may be solved. Such equations arise in well known ways, for example, in specifying both the syntax and semantics of programming languages. In many examples, the setting is some kind of ordered algebra A with the properties that A contains a least element $\bot$, and ω-chains, i.e., increasing sequences $a_0 \leq a_1 \leq \ldots$ have least upper bounds. In this setting, the least solution of an equation

$$x \;=\; f(x),$$

when $f : A \longrightarrow A$ preserves least upper bounds of ω-chains may be found as the least upper bound of

$$\bot \leq f(\bot) \leq f^2(\bot) \leq \ldots$$

For one such example, if Σ is a *ranked alphabet*, i.e., a sequence Σ_n, $n \geq 0$, of pairwise disjoint sets, the collection of finite and infinite Σ-trees may be equipped with an ordering by adjoining a new label $\bot$ to Σ_0, and defining $s \leq t$ if t may be obtained from s by adjoining some trees to leaves of s labeled $\bot$ (see below, or [GTWW77, Gue81] and [BE93] for example).

Similarly, the category of all partial functions $X \longrightarrow X$ is naturally ordered by set-inclusion of graphs; then, the meaning of a looping construct, such as the `while-do`:

```
(while B? f)(x) =
    if (B? x) then (while B? f) (f x)
    else x
```

is the least upper bound of the sequence $f_0, f_1, \ldots,$ of partial functions, where $f_0 : X \longrightarrow X$ is the totally undefined function, and

$$f_{n+1}(x) \;=\; \texttt{if}\ (B?\ x)\ \texttt{then}\ (f_n(f(x))\ \texttt{else}\ x.$$

However, not all fixed point equations may be solved by means of least upper bounds. One example that plays an important role in the semantics of parallel computation is **synchronization trees**, see [Mil89, Win84][1] or [BE93]. For a fixed alphabet A, an A-synchronization tree is a finite or countable rooted tree, in which every edge is labeled by a letter in A; the collection of these trees forms a category $\mathcal{ST}_A$, in which a morphism $f : s \longrightarrow t$ is a function from the vertices of s to the vertices of t which preserves the root, the edge relation and the labeling. This category has an initial object $\perp$, the rooted tree with no edge, and is equipped with at least the operations of **prefixing** and **sum**. For each letter $a \in A$, and each synchronization tree t, $a : t$ is the tree obtained from t by adding a new root, r and an edge labeled a from r to the root of t. When s, t are synchronization trees, $s + t$ is the tree obtained from s, t by identifying their roots, and otherwise, keeping the vertices and edges of each. In this category, fixed point equations such as

$$x \quad = \quad (a : x) + x$$

have solutions, but there is no canonical ordering on the category in which least solutions exist. However, this category has all colimits of ω-diagrams; the right-side of fixed point equations determines a continuous endofunctor $F : \mathcal{ST}_A \longrightarrow \mathcal{ST}_A$. Further, the "initial fixed point" of the functor F is determined up to isomorphism as a colimit of the ω-diagram

$$\perp \xrightarrow{!} F(\perp) \xrightarrow{F(!)} F^2(\perp) \xrightarrow{F^2(!)} \dots .$$

Note that $C = \mathcal{ST}_A$ is a category, equipped with several functors:

$$C^2 \xrightarrow{+} C, \;\; C \xrightarrow{a:} C, \;\; a \in A.$$

All of these functors are continuous.

Thus, labeled trees, partial functions and synchronization trees are examples of (continuous) **categorical Σ-algebras**, or cΣa, namely categories equipped with functors from powers of the category to itself. There are other examples which we will mention after stating our main results.

[1] In [Win84], two complete partial orders are defined on synchronization trees. However, the definition depends on the concrete representation of trees and is thus not fully abstract.

Not all categorical algebras enjoy the property that colimits of all ω-diagrams exist. For example, the cΣa of finite trees, and the cΣa of finite synchronization trees lacks such completeness. We present an elementary proof that any cΣa may be embedded in an ω-complete one in a canonical way.

Our completion result is a generalization of the one for ordered algebras, given in Bloom [Blo76]. There is a "filtered" completion result for categories (without additional algebraic structure) in Johnstone [Ele02], volume 2. It took us a while to see that Johnstone's construction is indeed a generalization of ours, but he is not concerned with the algebraic structure, equations and inequations; furthermore, his proof is intended for experts is category theory. We give an alternate proof of this result, which is perhaps more accessible to theoretical computer scientists. Our result shows that the algebraic structure of a cΣa C extends, in essentially only one way, to the completion C^ω. As in [Blo76], we show that the equations and inequations that hold in C^ω are exactly those that hold in C.

The notion of a cΣa probably occurs to all those familiar with both universal algebra and category theory, and the outline of an ω-completion result is probably obvious to many. Perhaps the "right" notion of the truth of an inequation in a cΣa is not obvious, and the details of the construction have turned out to be more delicate than expected. We think they merit exposition in this paper.

2 Some notation

$\mathbb{N}$ is the category whose objects are the nonnegative integers, in which there is a morphism $n \longrightarrow p$ exactly when $n \leq p$. If $f : X \longrightarrow Y$ is either a function or functor, we write

$$i\,f,\;\; f_i,\;\; f(i)$$

for the value of f on the argument i. The **composite** of $f : x \longrightarrow y$ and $g : y \longrightarrow z$ is written $fg : x \longrightarrow z$ or $f \cdot g$, where f, g are functions or functors. If $f : \mathbb{N} \longrightarrow C$ is a functor, a cocone over f is sometimes written $(\nu_n : f_n \longrightarrow d)_n$.

3 The completion and characterization theorems

Let Σ be a ranked alphabet, i.e., $\Sigma = \cup_{n\geq 0}\Sigma_n$ is the disjoint union of sets Σ_n.

Definition 3.1 *A* ***categorical*** Σ***-algebra (c***Σ***a for short)*** *C consists of a small category C, and, for each letter $\sigma \in \Sigma_n$, a functor*

$$\sigma_C : C^n \longrightarrow C.$$

*A cΣa-**morphism** $h : C \longrightarrow D$ of categorical Σ-algebras is a functor $h : C \longrightarrow D$ such that for each $n \geq 0$ and each $\sigma \in \Sigma_n$,*

$$\sigma_C \cdot h = C^n \xrightarrow{\sigma_C} C \xrightarrow{h} D$$

and

$$h^n \cdot \sigma_D = C^n \xrightarrow{h^n} D^n \xrightarrow{\sigma_D} D$$

*are naturally isomorphic. A cΣa-morphism h is **strict** if the functors $\sigma_C \cdot h$ and $h^n \cdot \sigma_D$ are the same, for all $\sigma \in \Sigma_n$. If C, D are cΣa's, and C is a subcategory of D, then C is a **sub-cΣa** of D if the inclusion*

$$C \xrightarrow{\mathsf{inc}} D$$

is a cΣa-morphism.

Recall that a functor $h : D \longrightarrow D'$ is **ω-continuous**, or just "continuous", for short, if whenever a functor $f : \mathbb{N} \longrightarrow D$ has a colimit $(\nu_n : f_n \longrightarrow d)_n$ in D, then $(\nu_n h : f_n h \longrightarrow dh)_n$ is a colimit of $f \cdot h : \mathbb{N} \longrightarrow D'$.

A cΣa C is **(ω-)continuous** if

- C is ω-complete, i.e., C has an initial object and all functors $\mathbb{N} \longrightarrow C$ have colimits, and further,
- for each $n \geq 0$ and $\sigma \in \Sigma_n$, the functor $\sigma_C : C^n \longrightarrow C$ is continuous.

A (strict) **morphism of continuous cΣa's** $F : C \longrightarrow D$ is a continuous functor F which is a (strict) cΣa morphism.

Remark 3.2 *Categorical Σ-algebras are a generalization of ordered Σ-algebras and continuous cΣa's are a generalization of (order) continuous Σ-algebras, see [Blo76, GTWW77, Gue81] or below.*

Let $\mathbf{Tm}_\Sigma(p)$ denote the collection of Σ-terms on p variables $x_1, \ldots, x_p$. Suppose that C is a cΣa. Any term $t \in \mathbf{Tm}_\Sigma(p)$ determines a functor $t_C : C^p \longrightarrow C$ as follows:

- $(x_i)_C : C^p \longrightarrow C$ is the i-th projection functor $(1 \leq i \leq p)$.
- If $\sigma \in \Sigma_k$, $0 \leq k$, $(\sigma(t_1, \ldots, t_k))_C$ is the composite

$$C^p \xrightarrow{\langle (t_1)_C, \ldots, (t_k)_C \rangle} C^k \xrightarrow{\quad \sigma_C \quad} C$$

A cΣa inequality is an expression

$$s \preceq t$$

where s, t are terms in $\mathbf{Tm}_\Sigma(p)$, for some $p \geq 0$. If C is a cΣa, we say C **is a model for** $s \preceq t$, in symbols,

$$C \models s \preceq t$$

if there is a natural transformation $s_C \longrightarrow t_C$ between the functors s_C and t_C. Similarly, we define a **cΣa equality** to be an expression $s \cong t$, where s, t are as before. We write

$$C \models s \cong t$$

if there is a natural **isomorphism** $s_C \longrightarrow t_C$.

Definition 3.3 *Suppose that C and D are cΣa's, and D is continuous. Suppose further that $\eta : C \longrightarrow D$ is a cΣa-morphism. We say (η, D)* ***is the ω-completion of*** *C if, for any continuous cΣa D', and any cΣa morphism $F : C \longrightarrow D'$ there is a morphism $F^\omega : D \longrightarrow D'$ in the category of continuous cΣa's, unique up to a natural isomorphism, such that the diagram*

$$\begin{array}{ccc} C & \xrightarrow{\eta} & D \\ & {\scriptstyle F}\searrow & \downarrow {\scriptstyle F^\omega} \\ & & D' \end{array}$$

commutes. If F preserves initial objects, so does F^ω.

When $\eta : C \longrightarrow D$ is an inclusion, we say only "D is the ω-completion of C", even though D is only determined up to equivalence.

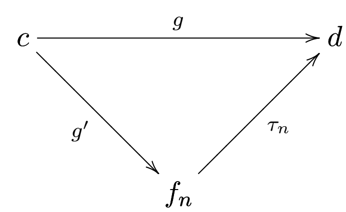
Our characterization of the ω-completion involves the following notion.

Definition 3.4 *Suppose that C is a full subcategory of D. D is* ***compactly generated by*** *C if*

- ***(factorization property)*** *For any object c in C, and any functor $f : \mathbb{N} \longrightarrow C$, if*

  $$(\tau_i : f_i \longrightarrow d)_i \tag{1}$$

  is a colimiting cone in D, then each map $g : c \longrightarrow d$ factors through some τ_n, i.e., there is some $n \geq 0$ and some $g' : c \longrightarrow f_n$ such that

  commutes.

- ***(completeness property)*** *For each object d in D there is a functor $f : \mathbb{N} \longrightarrow C$ and a colimiting cone with vertex d as in (1).*

- ***(colimits are finitary)*** *If $f : \mathbb{N} \longrightarrow C$ is a functor with colimiting cone (1), then for each $n \geq 0$, if $g, g' : c \longrightarrow f_n$ are maps in C such that*

  $$g \cdot \tau_n \;=\; g' \cdot \tau_n,$$

  then there is some $p > n$ such that

  $$g \cdot f(n,p) \;=\; g' \cdot f(n,p).$$

For example if D is the category of finite or countable sets, and C is the category of finite sets, then D is compactly generated by C.

We have the following main results.

Theorem 3.1 (Characterization Theorem) *Suppose that C is a cΣa, and $\eta : C \longrightarrow D$ is a functor which is is full, faithful, injective on objects, and preserves initial objects. Then if D is compactly generated by $\eta(C)$, D admits the structure of a continuous cΣa in essentially one way such that (η, D) is the ω-completion of C.*

Theorem 3.2 (Completion theorem) *Any cΣa C has an ω-completion $\eta_\omega : C \longrightarrow C^\omega$, and η_ω is full, faithful, injective on objects, preserves initial objects, C^ω is compactly generated by $\eta_\omega(C)$.*

It follows that

- *(η, D) is an ω-completion of C iff the conditions in the Characterization Theorem hold, since C^ω is unique up to categorical equivalence.*
- *Any cΣa inequality or equality which holds in C, also holds in C^ω.*

The proofs of these theorems will be given after the discussion of some examples.

A Correction

In an extended abstract of this paper, [BE06] our characterization theorem contained an error. We incorrectly defined the notion of compact generation. In place of the "finitary colimits" condition above, we said that for each functor $f : \mathbb{N} \longrightarrow C$, and any colimit diagram

$$(\tau_i : f_i \longrightarrow d)_i,$$

each morphism τ_i is monic. Of course, this condition implies finitary colimits, but it is certainly not equivalent to it.

3.1 Ordered Σ-algebras

When Σ is a ranked set, an **ordered Σ-algebra** consists of a partially ordered set $(A, \leq)$ equipped with a function

$$\sigma : A^n \longrightarrow A$$

which is order preserving. Such algebras are categorical Σ-algebras, in which the objects are the elements of A and in which there is a morphism $a \longrightarrow b$ exactly when $a \leq b$. Also, when s, t are in $\mathbf{Tm}_\Sigma(p)$, an inequation $s \leq t$ holds in A exactly when there is a natural transformation $s \longrightarrow t$.

In [Blo76], varieties of ordered algebras were considered, and it was shown that each variety V was closed under the free ω-completion of any algebra in V. Our main theorem is a significant generalization of this result.

3.2 Σ-trees

As formalized in [BET93], a Σ-tree t is a partial function $t : \mathbb{N}_+^* \longrightarrow \Sigma$, with source the set $\mathbb{N}_+^*$ of finite sequences of positive integers, and target Σ, with the following properties.

- The domain of t is a nonempty, prefix-closed subset of $\mathbb{N}_+^*$.
- If $u \in \mathbb{N}_+^*$ is in the domain of t and if $t(u) \in \Sigma_n$, and i is a positive integer, then ui, the sequence obtained by putting i at the end of the sequence u, is in the domain of t iff $1 \leq i \leq n$. Thus, the leaves of t are those sequences u such that $t(u) \in \Sigma_0$.

We assume there is a distinguished letter $\perp \in \Sigma_0$. Then for trees s, t, we define $s \leq t$ if t may be obtained from s by attaching some trees to leaves of s labeled $\perp$. The collection $\mathbf{\Sigma tr}$ of Σ-trees is an ordered Σ-algebra, in which the letter $\sigma \in \Sigma_n$ denotes the "prefixing operation" which applied to the n-tuple of trees $(t_1, \ldots, t_n)$ produces the tree $\sigma(t_1, \ldots, t_n)$, with a new root labeled σ, whose immediate successors are the roots of $t_1, \ldots, t_n$, respectively. As a function, for $u, v \in \mathbb{N}_+^*$ and $i \in \mathbb{N}_+$,

$$\sigma(t_1, \ldots, t_n)(u) \quad = \quad \begin{cases} \sigma & \text{if } u \text{ is the empty sequence} \\ t_i(v) & \text{if } u = iv, \end{cases}$$

where iv is the sequence obtained by putting i on the front of the sequence v. $\Sigma\mathbf{tr}$ is an **ordered** cΣa, in that there is a morphism $s \longrightarrow t$, for any trees s, t iff $s \leq t$. It is well known that $\Sigma\mathbf{tr}$ is a continuous cΣa.

Let $\Sigma\mathbf{Ftr}$ denote the full subcategory of $\Sigma\mathbf{tr}$ determined by the finite trees (those whose domain is finite). Then, by Corollary 5.13,

Proposition 3.5 $\Sigma\mathbf{tr}$ *is the completion of* $\Sigma\mathbf{Ftr}$. □

Note that if D is any cΣa with an initial object $\perp_D$, there is a unique cΣa morphism $\Sigma\mathbf{Ftr} \longrightarrow D$ taking $\perp$ to $\perp_D$. Thus,

Corollary 3.6 $\Sigma\mathbf{tr}$ *is the initial continuous cΣa in the category of all continuous cΣa's in which $\perp$ is the initial object: for any such continuous cΣa D there is a continuous cΣa-morphism $\Sigma\mathbf{tr} \longrightarrow D$, unique up to an isomorphism.* □

3.3 Synchronization trees

We have shown in [BE93] that $\mathcal{ST}_A$ defined briefly in the introduction is an ω-continuous categorical Σ_A-algebra, where Σ is the signature having a constant symbol 0, denoting the initial object $\perp$, a unary function symbol a for each $a \in A$, denoting the prefixing operation, and a binary function symbol $+$, denoting the coproduct operation described above. See also [Mil89, Win84].

Let $\mathcal{FST}_A$ denote the full subcategory of $\mathcal{ST}_A$ determined by the finite synchronization trees. Note that $\mathcal{FST}_A$ is also a cΣa, a "categorical subalgebra" of $\mathcal{ST}_A$. By Corollary 5.13, we have

Proposition 3.7 $\mathcal{ST}_A$ *is the completion of* $\mathcal{FST}_A$. □

Let $\mathcal{V}$ be the collection of all cΣa's D in which 0 is an initial object which satisfy the following:

$$\begin{aligned} x + 0 &\cong x \\ x + y &\cong y + x \\ x + (y + z) &\cong (x + y) + z \end{aligned}$$

Then it is not hard to show that the subcategory $\mathcal{FST}_A(\text{mon})$ of $\mathcal{FST}_A$ with the same objects having only monics as morphisms is the initial cΣa in $\mathcal{V}$, in the following sense: for any cΣa in $\mathcal{V}$ there is a cΣa-morphism $F : \mathcal{FST}_A(\text{monics}) \longrightarrow D$, unique up to a natural isomorphism.

Corollary 3.8 *$\mathcal{FST}_A(mon)^\omega$ is initial in the category of all continuous cΣa's in $\mathcal{V}$.*

Proof. Let D be a continuous cΣa in $\mathcal{V}$. Then there is a cΣa morphism $F : \mathcal{FST}_A(\text{mon}) \longrightarrow D$, since $\mathcal{FST}_A(\text{mon})$ is initial in D. But then there is a continuous $F^\omega : \mathcal{FST}_A(\text{mon})^\omega \longrightarrow D$, unique up to natural isomorphism, by the completion theorem. □

3.4 Words

We recall from [Cou78, BE05] that when A is a finite or countable set, a **word over** A (called an *arrangement* in [Cou78]) is a triple $u = (L_u, \leq_u, \lambda_u)$ consisting of a finite or countable linearly ordered set $(L_u, \leq_u)$ and a labeling function $\lambda : L_u \longrightarrow A$. A word u is finite if the set L_u is finite. A morphism between words $u = (L_u, \leq, \lambda_u)$ and $v = (L_v, \leq, \lambda_v)$ is an order and label preserving map $h : L_u \longrightarrow L_v$. It is clear that words over A and their morphisms form a category that we denote $\mathcal{W}_A$. The finite words over A determine a full subcategory of $\mathcal{W}_A$ denoted $\mathcal{FW}_A$.

The basic operation on words is **concatenation** $u, v \mapsto u; v$ defined as follows. Without loss of generality we may assume that L_u and L_v are disjoint. Then the *concatenation* $u \cdot v$ is the word whose underlying linear order is $(L_u \cup L_v, \leq)$ where $x \leq y$ for all $x \in L_u$ and $y \in L_v$ and such that the restriction of $\leq$ to L_u agrees with $\leq_u$ and the restriction of $\leq$ to L_v agrees with $\leq_v$. The labeling function λ is given by

$$\lambda(x) \quad = \quad \begin{cases} \lambda_u(x) & \text{if } x \in L_u \\ \lambda_v(x) & \text{if } x \in L_u. \end{cases}$$

We extend concatenation to a functor. Given $f : u \longrightarrow u'$ and $g : v \longrightarrow v'$, we define the morphism $f \cdot g : u \cdot v \longrightarrow u' \cdot v'$ so that it agrees with f on the elements of L_u and with g on the elements of L_v.

Let Σ be the signature with a constant symbol a, for each $a \in A$, denoting the constant functor $\mathcal{W}_A^0 \longrightarrow \mathcal{W}_A$ whose value is the singleton word labeled a, a symbol 0 in Σ_0 denoting the constant functor whose value is the empty word, and a binary function symbol ; denoting the concatenation functor. The following fact was essentially shown in [Cou78].

Proposition 3.9 *$\mathcal{W}_A$ is a continuous cΣa.*

In $\mathcal{W}_A$, one can solve such equations as $x = a; x$ and $x = x; a; x$. The initial solution to the second, is the word $[\![a]\!]^\eta$ whose underlying order is isomorphic to the rationals, with every point labeled a. (There doesn't seem to be an ordering of $\mathcal{W}_A$ such that $[\![a]\!]^\eta$ is the least upper bound of a sequence of finite approximations.)

Let $\mathcal{FW}_A$ be the full subcategory of $\mathcal{W}_A$ determined by the finite words. It follows from Corollary 5.13,

Proposition 3.10 *$\mathcal{W}_A$ is the completion of $\mathcal{FW}_A$.*

Let $\mathcal{FW}_A$(mon) be the subcategory of $\mathcal{FW}_A$ with the same objects, having only the monics as morphisms. Define the category $\mathcal{M}$ having as objects all cΣa's with an initial object 0 which satisfy the monoid equations

$$\begin{aligned} 0; x &\cong x \\ x; 0 &\cong x \\ x; (y; z) &\cong (x; y); z \end{aligned}$$

It is not hard to show that $\mathcal{FW}_A$(mon) is freely generated by A in $\mathcal{M}$ in the sense that for any cΣa D in $\mathcal{M}$, and any function $f : A \longrightarrow \text{obj}(D)$, mapping 'letters' in A to objects in D, there is a functor $F : \mathcal{FW}_A(\text{mon}) \longrightarrow D$, unique up to a natural isomorphism, such that $F(a) = f(a)$, for each $a \in A$. Thus,

Corollary 3.11 *$\mathcal{FW}_A(mon)^\omega$ is freely generated by A in the category of all continuous cΣa's in $\mathcal{M}$.* □

4 Weak maps

An endofunctor $m : \mathbb{N} \longrightarrow \mathbb{N}$ is just a nondecreasing function. We say an endofunctor m is **unbounded** if for each $i \in \mathbb{N}$, there is some $j \in \mathbb{N}$ such that $i \leq jm$.

When $m : \mathbb{N} \longrightarrow \mathbb{N}$ is an endofunctor and $f : \mathbb{N} \longrightarrow C$ is a chain, we write mf for the composite

$$\mathbb{N} \xrightarrow{m} \mathbb{N} \xrightarrow{f} C.$$

Thus, for $i \in \mathbb{N}$, $(mf)_i = f_{im}$.

When f, g are chains, a **weak map** $\alpha : f \longrightarrow g$ is a natural transformation

$$\alpha : f \longrightarrow m_\alpha g$$

for some unbounded endofunctor m_α on $\mathbb{N}$. We define the composite of weak maps $\alpha : f \longrightarrow g$ and $\beta : g \longrightarrow h$ as

$$\alpha \circ \beta \quad := \quad f \xrightarrow{\alpha} m_\alpha g \xrightarrow{m_\alpha \beta} (m_\alpha m_\beta) h.$$

(When needed to specify the endofunctor, we will denote a weak map α as $\alpha : f \longrightarrow m_\alpha g$.)

It is an easy exercise to prove that composition of weak maps is associative (where defined), and thus we have a category

$$W(C) \tag{2}$$

whose objects are the functors $\mathbb{N} \longrightarrow C$, and whose morphisms are the weak maps.

4.1 The equivalence on weak maps

We will make use of the following family of equivalence relations on weak maps.

Definition 4.1 *For weak maps $\alpha : f \longrightarrow m_\alpha g$ and $\beta : f \longrightarrow m_\beta g$, we say*

$$\alpha \simeq \beta$$

if for all $i \geq 0$ there is some $j \geq im_\alpha, im_\beta$ such that

$$\alpha_i \cdot g(im_\alpha, j) \quad = \quad \beta_i \cdot g(im_\beta, j).$$

It follows that for all $k \geq j$,

$$\alpha_i \cdot g(im_\alpha, k) \quad = \quad \beta_i \cdot g(im_\beta, k).$$

It is clear that $\simeq$ is an equivalence relation on the weak maps with the same source and target. Let $[\alpha] : f \longrightarrow g$ denote the $\simeq$-equivalence class of the weak map $\alpha : f \longrightarrow g$. This equivalence relation is compatible with composition.

Proposition 4.2 *If* $\alpha \simeq \alpha' : f \longrightarrow g$ *and* $\beta \simeq \beta' : g \longrightarrow h$*, then* $\alpha \circ \beta \simeq \alpha' \circ \beta'$.

Proof. First, we show that if $\gamma : g \longrightarrow m_\gamma h$ is a weak map, then $\alpha \circ \gamma \simeq \beta \circ \gamma$.

Fix i, and choose $j \geq im_\alpha, im_\beta$ such that

$$\alpha_i \cdot g(im_\alpha, j) \quad = \quad \beta_i \cdot g(im_\beta, j).$$

Then

$$\begin{aligned} (\alpha \circ \gamma)_i \cdot h(im_\alpha m_\gamma, jm_\gamma) &= \alpha_i \cdot \gamma_{im_\alpha} \cdot h(im_\alpha m_\beta, jm_\gamma) \\ &= \alpha_i \cdot g(im_\alpha, j) \cdot \gamma_j \\ &= \beta_i \cdot g(im_\beta, j) \cdot \gamma_j \\ &= (\beta \circ \gamma)_i \cdot h(im_\beta m_\gamma, jm_\gamma). \end{aligned}$$

Thus, for all $k \geq jm_\gamma$,

$$(\alpha \circ \gamma)_i \cdot h(im_\alpha m_\gamma, k) \quad = \quad (\beta \circ \gamma)_i \cdot h(im_\beta m_\gamma, k).$$

We omit the proof that if $\gamma : h \longrightarrow m_\gamma f$, then

$$\gamma \circ \alpha \quad \simeq \quad \gamma \circ \beta$$

The proof is complete. □

We will need the following fact about the relation $\simeq$.

Lemma 4.3 (inflation lemma) *Suppose that* $\alpha : f \longrightarrow mg$ *and that* $m' : \mathbb{N} \longrightarrow \mathbb{N}$ *is any functor satisfying*

$$k\,m \quad \leq \quad km',$$

for all $k \geq 0$*. Define the natural transformation*

$$\alpha' : f \longrightarrow m'g$$

by

$$\alpha'_i \quad := \quad f_i \xrightarrow{\alpha_i} g_{im} \xrightarrow{g(im,\ im')} g_{im'}.$$

Then

$$\alpha \quad \simeq \quad \alpha'. \quad \square$$

Remark. We note without proof that for weak maps $\alpha : f \longrightarrow m_\alpha g, \beta : f \longrightarrow m_\beta g$, the following are equivalent.

- $\alpha \simeq \beta$
- for all $i \geq 0$ there are $j, j' \geq i$ such that $im_\alpha \leq jm_\beta$ and $im_\beta \leq j'm_\alpha$, and

$$\begin{aligned} f(i,\ j) \cdot \beta_j &= \alpha_i \cdot g(im_\alpha,\ jm_\beta) \\ f(i,\ j') \cdot \alpha_{j'} &= \beta_i \cdot g(im_\beta,\ j'm_\alpha). \end{aligned}$$

- There is some γ with $\alpha \sim \gamma$ and $\gamma \sim \beta$, where, for weak maps $\gamma : f \longrightarrow m_\gamma g$, $\delta : f \longrightarrow m_\delta g$,

$$\gamma \sim \delta$$

  if for all $i \geq 0$,

$$\begin{aligned} im_\gamma \leq im_\delta &\implies \delta_i = \gamma_i \cdot g(im_\gamma, im_\delta) \\ im_\delta \leq im_\gamma &\implies \gamma_i = \delta_i \cdot g(im_\delta, im_\gamma). \end{aligned}$$

- α, β are related by the least equivalence relation containing $\sim$.

Definition 4.4 *Recall the definition of the category $W(C)$ above. We define $W^{\simeq}(C)$ as the category whose objects are all functors $\mathbb{N} \longrightarrow C$; a morphism $[\alpha] : f \longrightarrow g$ is an $\simeq$-equivalence class of a weak map $\alpha : f \longrightarrow g$. Composition is well-defined by:*

$$[\alpha] \cdot [\beta] \quad := \quad [\alpha \circ \beta].$$

Lemma 4.5 (κ is a functor) *Suppose that C is a subcategory of D and for each functor*

$$f : \mathbb{N} \quad \longrightarrow \quad C$$

there is a colimiting cone

$$(\tau_i^f : f_i \quad \longrightarrow \quad \kappa(f))_i$$

in D. Using these cones, we may extend κ to a functor

$$\kappa : W(C) \quad \longrightarrow \quad D$$

by defining $\kappa(\alpha)$, for any weak map $\alpha : f \longrightarrow m_\alpha g$, as the unique map

$$\kappa(\alpha) : \kappa(f) \longrightarrow \kappa(g)$$

such that

$$\tau_i^f \cdot \kappa(\alpha) = \alpha_i \cdot \tau_{im_\alpha}^g,$$

for all $i \geq 0$.

Proof. We need prove only that κ preserves composition, since it is clear that it preserves the identity weak maps. So suppose that $\alpha : f \longrightarrow m_\alpha g$ and $\beta : g \longrightarrow m_\beta h$ are weak maps, where $f, g, h : \mathbb{N} \longrightarrow C$ are functors. By definition, $\kappa(\alpha \circ \beta)$ is the unique map such that

$$\tau_i^f \cdot \kappa(\alpha \circ \beta) = (\alpha \circ \beta)_i \cdot \tau_{im_\alpha m_\beta}^h.$$

But, for any i,

$$\begin{aligned}(\alpha \circ \beta)_i \cdot \tau_{im_\alpha m_\beta}^h &= \alpha_i \cdot (\beta_{im_\alpha} \cdot \tau_{im_\alpha m_\beta}^h) \\ &= (\alpha_i \cdot \tau_{im_\alpha}^g) \cdot \kappa(\beta) \\ &= \tau_i^f \cdot \kappa(\alpha) \cdot \kappa(\beta).\end{aligned}$$

Thus,

$$\kappa(\alpha \circ \beta) = \kappa(\alpha) \cdot \kappa(\beta). \quad \square$$

Lemma 4.6 (equivalence lemma) *Suppose that $f, f' : \mathbb{N} \longrightarrow C$ a functors. If $\gamma : f \longrightarrow mf'$ and $\overline{\gamma} : f \longrightarrow \overline{m}f'$ are weak maps such that $\gamma \simeq \gamma'$, then*

$$\kappa(\gamma) = \kappa(\overline{\gamma}).$$

Thus, κ becomes a functor $W^{\simeq}(C) \longrightarrow D$.

Proof. If $\gamma \simeq \overline{\gamma}$, we show for each $i \geq 0$,

$$\tau_i^f \cdot \kappa(\gamma) = \tau_i^f \cdot \kappa(\overline{\gamma}).$$

Indeed, for fixed i, let $j \geq im, i\overline{m}$ be such that

$$\gamma_i \cdot f'(im, j) = \overline{\gamma}_i \cdot f'(i\overline{m}, j).$$

Then

$$\begin{aligned}
\tau_i^f \cdot \kappa(\gamma) &= \gamma_i \cdot \tau_{im}^{f'} \\
&= \gamma_i \cdot f'(im, j) \cdot \tau_j^{f'} \\
&= \overline{\gamma}_i \cdot f'(i\overline{m}, j) \cdot \tau_j^{f'} \\
&= \overline{\gamma}_i \cdot \tau_{i\overline{m}}^{f'} \\
&= \tau_i^f \cdot \kappa(\overline{\gamma}),
\end{aligned}$$

so that $\kappa(\gamma) = \kappa(\overline{\gamma})$. □

Now, we give a condition sufficient to obtain a colimit of a functor $G : \mathbb{N} \longrightarrow D$.

Lemma 4.7 (Diagonal colimit lemma) *We assume the following hypotheses.*

- *For $i \geq 0$, $f^i : \mathbb{N} \longrightarrow D$ is a functor with colimiting cone*

$$(\tau_j^i : f_j^i \longrightarrow \kappa(f^i))_j.$$

- *For each $i \leq j$, $\beta^{i,j} : f^i \longrightarrow f^j$ is a natural transformation such that $\beta^{i,i} = \mathbf{1}_{f^i}$ and, when $i \leq j \leq k$,*

$$\beta^{i,j} \cdot \beta^{j,k} = \beta^{i,k}.$$

  Thus,

$$G : \mathbb{N} \longrightarrow D$$

  is a functor, where $G_i = \kappa(f^i)$, and for all $0 \leq i \leq j$, $G(i,j) = \kappa(\beta^{i,j})$, defined in Lemma 4.5.

- *$g : \mathbb{N} \longrightarrow D$ is the diagonal functor, defined on objects by*

$$g_i = f_i^i,$$

  and on morphisms by

$$\begin{aligned}
g(i,j) &:= f^i(i,j) \cdot \beta_j^{i,j} \qquad (3) \\
&= \beta_i^{i,j} \cdot f^j(i,j).
\end{aligned}$$

- *Let $\mu_i : \mathbb{N} \longrightarrow \mathbb{N}$ be the endofunctor defined by*

$$\mu_i(j) \;\; := \;\; \max(i, j),$$

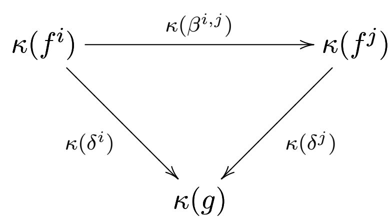
and let $\delta^i : f^i \longrightarrow \mu_i g$ be the weak map

$$\delta^i_j \;\; := \;\; \begin{cases} f^i(j, i) & j \leq i \\ \beta^{i,j}_j & i < j. \end{cases}$$

- *Suppose that $(\tau^g_i : g_i \longrightarrow \kappa(g))_i$ is a colimiting cone.*

Then, $(\kappa(\delta^i) : \kappa(f^i) \longrightarrow \kappa(g))_i$ is a colimiting cone over G, where $\kappa(\delta^i)$ is the functor defined in Lemma 4.5.

Proof. First, for $i \leq j$, the diagram



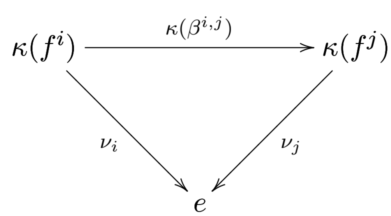
commutes, since by construction,

$$\delta^i \;\; \simeq \;\; \beta^{i,j} \cdot \delta^j.$$

Thus,

$$\begin{aligned} \kappa(\delta^i) &= \kappa(\beta^{i,j} \cdot \delta^j) \\ &= \kappa(\beta^{i,j}) \cdot \kappa(\delta^j). \end{aligned}$$

Now assume $(\nu_i : \kappa(f^i) \longrightarrow e)_i$ is any cone over G, so that for $i \leq j$,



commutes. Then, for each i, j,

$$\tau^i_j \cdot \nu_i \;\; = \;\; \delta^i_j \cdot \tau^g_{\mu_i(j)} \cdot \nu_{\mu_i(j)}. \tag{4}$$

We obtain a corresponding cone over g by letting

$$\overline{\nu}_i \quad := \quad g_i \xrightarrow{\tau_i^i} \kappa(f^i) \xrightarrow{\nu_i} e.$$

It is clear that $\overline{\nu}_i = g(i,j) \cdot \overline{\nu}_j$, for $i \leq j$. Thus, there is a unique $\nu^{\#} : \kappa(g) \longrightarrow e$ such that for all $i \geq 0$,

$$g_i \xrightarrow{\tau_i^i} \kappa(f^i) \xrightarrow{\nu_i} e \quad = \quad g_i \xrightarrow{\tau_i^g} \kappa(g) \xrightarrow{\nu^{\#}} e. \tag{5}$$

It follows from (4) that for each $i \geq 0$,

$$\nu_i \quad = \quad \kappa(\delta^i) \cdot \nu^{\#}.$$

Now suppose that $\theta : \kappa(g) \longrightarrow e$ is any morphism such that

$$\nu_i \quad = \quad \kappa(\delta^i) \cdot \theta, \tag{6}$$

for all $i \geq 0$. We must show $\theta = \nu^{\#}$. But, (6) implies that for all i, j,

$$\tau_j^i \cdot \nu_i \quad = \quad \delta_j^i \cdot \tau_{\mu_i(j)}^g \cdot \theta.$$

Letting $j = i$,

$$\tau_i^i \cdot \nu_i \quad = \quad \tau_i^g \cdot \alpha,$$

so that by (5), $\alpha = \nu^{\#}$. □

5 Compact generation and the Characterization Theorem

In this section, we prove the characterization theorem.

In the remainder of this section, unless specifically said otherwise, assume that D is a category compactly generated by the full subcategory C.

Lemma 5.1 (cone lemma) *Suppose that $f, g : \mathbb{N} \longrightarrow C$ are functors with colimiting cones in D:*

$$(\tau_i^f : f_i \longrightarrow d)_i \quad \textit{and} \quad (\tau_i^g : g_i \longrightarrow d')_i. \tag{7}$$

Suppose that $\varphi : f \longrightarrow d'$ is a cone over f in D, i.e., for each $i < j$, $\varphi_i : f_i \longrightarrow d$ is a map such that

$$\varphi_i \quad = \quad f(i,j) \cdot \varphi_j.$$

Then there is some weak map $\alpha : f \longrightarrow m_\alpha g$ such that for all i,

$$\varphi_i \quad = \quad \alpha_i \cdot \tau^g_{im_\alpha}.$$

Proof. We compute α_i by induction.

Suppose, using the factorization property, that for all integers i we have found an integer im_β and a map $\beta_i : f_i \longrightarrow g_{im_\beta}$ such that

$$\varphi_i \quad = \quad \beta_i \cdot \tau^g_{im_\beta},$$

and for $i < n$, we have found an integer $im_\alpha \geq im_\beta$ such that

$$\alpha_i \quad = \quad \beta_i \cdot g(im_\beta, im_\alpha)$$

and for $i < j < n$,

$$\alpha_i \cdot g(im_\alpha, jm_\alpha) \quad = \quad f(i,j) \cdot \alpha_j.$$

We show how to define nm_α and α_n.

If $n = 0$, let $\alpha_0 = \beta_0$. Otherwise, since φ is a cone over f, we have

$$f(n-1,n) \cdot \beta_n \cdot \tau^g_{nm_\beta} \quad = \quad \beta_{n-1} \cdot g((n-1)m_\beta, nm_\beta) \cdot \tau^g_{nm_\beta}.$$

Thus, for any $k > nm_\beta$,

$$f(n-1,n) \cdot \beta_n \cdot g(nm_\beta, k) \cdot \tau^g_k \quad = \quad \beta_{n-1} \cdot g((n-1)m_\beta, k) \cdot \tau^g_k.$$

In particular, for any $k \geq (n-1)m_\alpha, nm_\beta$,

$$\begin{aligned} f(n-1,n) \cdot \beta_n \cdot g(nm_\beta, k) \cdot \tau^g_k &= \beta_{n-1} \cdot g((n-1)m_\beta, k) \cdot \tau^g_k \\ &= \alpha_{n-1} \cdot g((n-1)m_\alpha, k) \cdot \tau^g_k. \end{aligned}$$

Using the fact that colimits are finitary, there is a $k' > (n-1)m_\alpha$ such that

$$f(n-1,n) \cdot \beta_n \cdot g(nm_\beta, k') \quad = \quad \alpha_{n-1} \cdot g((n-1)m_\alpha, k').$$

Define

$$\begin{aligned} nm_\alpha &= k' \\ \alpha_n &= \beta_n \cdot g(nm_\beta, nm_\alpha). \end{aligned}$$

This completes the induction. □

Definition 5.2 *Suppose that $f, g : \mathbb{N} \longrightarrow C$ are functors with colimiting cones as in (7). A* ***very weak map*** *$\beta : f \longrightarrow g$ is a collection of maps $(\beta_i : f_i \longrightarrow g_{im_\beta})_i$, where $m_\beta : \mathbb{N} \longrightarrow \mathbb{N}$ is an unbounded endofunctor, such that for each $i < j$,*

$$\beta_i \cdot g(im_\beta, jm_\beta) \cdot \tau^g_{jm_\beta} \quad = \quad f(i,j) \cdot \beta_j \cdot \tau^g_{jm_\beta}.$$

.

Note that if $\beta : f \longrightarrow g$ is a very weak map, there is a unique map

$$\kappa(\beta) : d \quad \longrightarrow \quad d'$$

such that for all i,

$$\tau^f_i \cdot \kappa(\beta) \quad = \quad \beta_i \cdot \tau^g_{im_\beta}.$$

Corollary 5.3 *Suppose that $\beta : f \longrightarrow g$ is a very weak map. Then there is a weak map $\alpha : f \longrightarrow g$ such that*

$$\kappa(\alpha) \quad = \quad \kappa(\beta). \quad \Box$$

Proof. In effect, this is what we proved in Lemma 5.1. $\Box$

Lemma 5.4 (κ is full) *Suppose $f, g : \mathbb{N} \longrightarrow C$ are functors with colimiting cones (7). If $h : d \longrightarrow d'$ is any map, there is some weak map $\alpha : f \longrightarrow g$ such that*

$$\kappa(\alpha) \quad = \quad h.$$

Proof. We need only show how to obtain a very weak map $\beta : f \longrightarrow g$ with $\kappa(\beta) = h$, and apply Corollary 5.3. Consider the maps

$$\gamma_i \quad := \quad \tau^f_i \cdot h : f_i \longrightarrow d'.$$

By the factorization property, there is some im_β and a map $\beta_i : f_i \longrightarrow g_{im_\beta}$ such that

$$\gamma_i \quad = \quad \beta_i \cdot \tau^g_{im_\beta}.$$

We may assume that for $i < j$, $im_\beta < jm_\beta$. Then, since

$$\gamma_i \quad = \quad f(i,j) \cdot \gamma_j,$$

we have

$$\beta_i \cdot g(im_\beta, jm_\beta) \cdot \tau^g_{jm_\beta} \quad = \quad f(i,j) \cdot \beta_j \cdot \tau^g_{jm_\beta}.$$

Thus, β is a very weak map such that

$$\kappa(\beta) \quad = \quad h.$$

This completes the proof. □

Lemma 5.5 (κ is faithful) *Suppose that $f, g : \mathbb{N} \longrightarrow C$ are functors with colimiting cones (7). If $\gamma, , \overline{\gamma} : f \longrightarrow g$ are weak maps with*

$$\kappa(\gamma) \quad = \quad \kappa(\overline{\gamma}).$$

then $\gamma \simeq \overline{\gamma}$.

Proof. Suppose that $\gamma : f \longrightarrow mg$ and $\overline{\gamma} : f \longrightarrow \overline{m}g$ are weak maps such that

$$\begin{aligned} \tau^f_i \cdot h &= \gamma_i \cdot \tau^g_{im} \\ &= \overline{\gamma}_i \cdot \tau^g_{i\overline{m}}, \end{aligned}$$

for all $i \geq 0$. We show $\gamma \simeq \overline{\gamma}$. Indeed, suppose for any fixed i, we choose $j \geq im, i\overline{m}$. Then, since τ^g is a cocone over g,

$$\begin{aligned} \gamma_i \cdot g(im, j) \cdot \tau^g_j &= \gamma_i \cdot \tau^g_{im} \\ &= \overline{\gamma}_i \cdot \tau^g_{i\overline{m}} \\ &= \overline{\gamma}_i \cdot g(i\overline{m}, j) \cdot \tau^g_j. \end{aligned}$$

Since the colimit τ^g is finitary, there is some $p > j$ such that

$$\gamma_i \cdot g(im, p) \quad = \quad \overline{\gamma}_i \cdot g(i\overline{m}, p),$$

showing $\gamma \simeq \overline{\gamma}$. □

Lemma 5.6 (two cone lemma) *Suppose that (7) is a pair of colimiting cones with the* ***same vertex*** $d = d'$. *Then there are weak maps $\alpha : f \longrightarrow g$ and $\beta : g \longrightarrow f$ such that*

$$\begin{aligned} \kappa(\alpha \circ \beta) &= \mathsf{id}_d \\ \kappa(\beta \circ \alpha) &= \mathsf{id}_d. \end{aligned}$$

Proof. Apply Lemma 5.4 to $\mathsf{id}_d : d \longrightarrow d$ in each direction. □

Lemma 5.7 (initial lemma) *An initial object in C is also initial in D.*

Proof. Suppose that $\bot_C$ is initial in C and d is any object in D. Let $f : \mathbb{N} \longrightarrow C$ be a functor with colimiting cone

$$(\tau_i : f_i \quad\longrightarrow\quad d)_i.$$

There is a unique map

$$u : \bot_C \quad\longrightarrow\quad f_0,$$

and thus

$$u_d \quad=\quad \bot_C \xrightarrow{u} f_0 \xrightarrow{\tau_0} d$$

is one map $\bot_C \longrightarrow d$. Let $g : \bot_C \longrightarrow d$ be any map. By the factorization property, there is some n and some $g' : \bot_C \longrightarrow f_n$ such that

$$g \quad=\quad g' \cdot \tau_n.$$

But

$$g' \quad=\quad u \cdot f(0, n),$$

by the initiality of $\bot_C$, so that

$$g \quad=\quad u_d. \quad \square$$

Lemma 5.8 (continuity lemma) *D has the following two properties.*

1. *D is ω-complete.*
2. *A functor $D \longrightarrow D'$ is continuous iff it preserves colimits of all functors $\mathbb{N} \longrightarrow C$.*

Proof of part 1. We know that each functor $\mathbb{N} \longrightarrow C$ has a colimit in D. We show that if $G : \mathbb{N} \longrightarrow D$ is a functor, G has a colimit in D.

Write the object G_n as $\kappa(f^n)$, where, for each $n \geq 0$, $f^n : \mathbb{N} \longrightarrow C$ be a functor such that $(\tau_i^n : f_i^n \longrightarrow G_n = \kappa(f^n))_i$ is a colimiting cone in D.

By Lemma 5.4, for each $0 \leq i \leq j$, each morphism $G(i,j) : G_i \longrightarrow G_j$ is determined by a weak map

$$\beta^{i,j} : f^i \quad \longrightarrow \quad m_{i,j} f^j.$$

For ease of notation, let's assume that all functors $m_{i,j}$ are the identity, so that for each $0 \leq i \leq j$, $\beta^{i,j} : f^i \longrightarrow f^j$ is a natural transformation, as in Lemma 4.7.

As before, define $g : \mathbb{N} \longrightarrow C$ as the diagonal functor (3).

Since every functor $\mathbb{N} \longrightarrow C$ has a colimit in D, let $(\tau_i^g : g_i \longrightarrow \kappa(g))_i$ be a colimit in D.

For each $i \geq 0$, there is a weak map $\delta^i : f^i \longrightarrow \mu_i g$ defined by

$$\delta_j^i \quad := \quad \begin{cases} f^i(j,i) & j \leq i \\ \beta_j^{i,j} & i < j. \end{cases}$$

We have shown in Lemma 4.7 that

$$(\kappa(\delta^i) : \kappa(f^i) \quad \longrightarrow \quad \kappa(g))_i$$

is a colimit of G $\square$

Proof of part 2. Suppose that $F : D \longrightarrow D'$ preserves the colimits of all functors $\mathbb{N} \longrightarrow C$. We show that F preserves the colimits of all functors $\mathbb{N} \longrightarrow D$. Indeed, suppose that $G : \mathbb{N} \longrightarrow D$ is a functor. Using the notation of the previous part, we have shown that

$$(\kappa(\delta^i) : G_i \quad \longrightarrow \quad \kappa(g))_i$$

is a colimit of G, where, for each $i \geq 0$, $f^i : \mathbb{N} \longrightarrow C$ is a functor and

$$(\tau_j^i : f_j^i \quad \longrightarrow \quad G_i)_j$$

is a colimit in D, and where g is the diagonal functor, with colimiting cone

$$(\tau_i^g : g_i \quad \longrightarrow \quad \kappa(g))_i.$$

But now, applying F, the assumptions imply that

$$(\tau^i_j F : f^i_j F \longrightarrow G_i F)_j$$

is a colimiting cone, as is

$$(\tau^g_i F : g_i F \longrightarrow \kappa(g) F)_i.$$

It then follows from Lemma 4.7 that

$$(\kappa(\delta^i) F : G_i F \longrightarrow gF)_i$$

is a colimiting cone in D'. □

It is easy to verify the following fact.

Lemma 5.9 (power lemma) *For each nonnegative integer n, D^n is compactly generated by C^n.* □

The next Lemma is a major step in the characterization theorem.

Lemma 5.10 (extension lemma) *If D' is ω-complete, and $F : C \longrightarrow D'$ is a functor, then there is a continuous $F^\# : D \longrightarrow D'$ such that*

$$F \quad = \quad C \xrightarrow{\mathsf{inc}} D \xrightarrow{F^\#} D',$$

at least up to natural isomorphism. Further, $F^\#$ itself is unique, up to natural isomorphism. If F preserves initial objects, so does $F^\#$.

Proof. Since D' is ω-complete, for each functor $f : \mathbb{N} \longrightarrow C$, we may choose a colimiting cone for the functor

$$\mathbb{N} \xrightarrow{f} C \xrightarrow{F} D',$$

say

$$(\lambda^f_i : f_i F \longrightarrow \kappa(fF))_i$$

in D'. We assume that when f is a constant functor, (i.e., for all $i \leq j$, $f_i = f_j$ and $f(i,j)$ is the corresponding identity morphism) then each morphism λ^f_i is the corresponding identity map. Since D is compactly generated by C, for each object d in D, choose a functor $f : \mathbb{N} \longrightarrow C$ and a colimiting cone

$$(\tau_i^f : f_i \longrightarrow d = \kappa(f))_i$$

in D. (When d is in fact an object in C, we assume $\tau_i^f = \mathsf{id}_d$, for each $i \geq 0$.) By Lemma 5.4, each arrow in D is $\kappa(\alpha)$ for a weak map α. For functors $f, g : \mathbb{N} \longrightarrow C$, and each arrow $h : \kappa(f) \longrightarrow \kappa(g)$ in D, choose a weak map $\alpha : f \longrightarrow g$ such that $\kappa(\alpha) = h$. Note that applying the functor F to each arrow of α, we obtain a weak map $\alpha F : fF \longrightarrow gF$ in D'.

We now define the extension $F^{\#}$ as follows:

$$\begin{aligned} F^{\#}(\kappa(f)) &:= \kappa(fF) \\ F^{\#}(\kappa(\alpha)) &:= \kappa(\alpha F). \end{aligned}$$

It is clear then that $F^{\#}$ extends F and preserves at least some ω-colimits. In order to show $F^{\#}$ preserves all ω-colimits, it is sufficient to show $F^{\#}$ preserves the colimits of all functors $f : \mathbb{N} \longrightarrow C$, by Lemma 5.4 of the previous draft. So suppose that $g : \mathbb{N} \longrightarrow C$, and

$$(\nu_i^g : g_i \longrightarrow d) \tag{8}$$

is any colimiting cone with the same vertex. We show that the image under $F^{\#}$ of this cone is a colimiting cone. But by Lemma 5.6, there are weak maps $\alpha : f \longrightarrow g$ and $\beta : g \longrightarrow f$ with

$$\begin{aligned} \kappa(\alpha \circ \beta) &= \mathsf{id}_d \\ \kappa(\beta \circ \alpha) &= \mathsf{id}_d. \end{aligned}$$

Applying F to these weak maps, we see that the F-image of the cone (8) is a colimit also. Since $F^{\#}$ preserves the colimit of f, it preserves the colimit of g.

Now if $\bot_C$ is initial in C, it is initial in D also, by Lemma 5.7. Thus, if F preserves initial objects, then $F^{\#}(\bot_C) = F(\bot_C)$ is initial in D'. Since $F^{\#}$ is defined by a particular choice of colimiting cones in D', it is unique up only to a natural isomorphism.

Since we have chosen constant functors $\mathbb{N} \longrightarrow C$ whose limit is an object of C, by the choice of the colimits λ, we see $F^{\#}$ extends F.

To show that $F^{\#}$ preserves compostion, we use the fact that $\kappa(\alpha \circ \beta) = \kappa(\alpha) \cdot \kappa(\beta)$, as shown in the previous version. Last, it is clear that $F^{\#}(\mathsf{id}_d) = \mathsf{id}_{F^{\#}(d)}$.

The proof is complete. $\square$

What about the Σ-functors?

Proposition 5.11 (Σ-structure extension) *Suppose that D is compactly generated by C and that C is a cΣa. There is an essentially unique way to impose the structure of a cΣa on D such that D becomes a continuous cΣa and $\mathsf{inc} : C \longrightarrow D$ is a cΣa-morphism.*

Proof. For a letter $\sigma \in \Sigma_n$, consider the functor

$$H_\sigma \quad = \quad C^n \xrightarrow{\sigma_C} C \xrightarrow{\mathsf{inc}} D.$$

Using Lemma 5.9 and the Extension Lemma 5.10, we may define σ_D as the extension $H^{\#}_{\Sigma}$ of H_σ. The rest is clear. $\square$

Corollary 5.12 ($F^{\#}$ is cΣa morphism) *Assume C, D, D' are cΣa's, D is compactly generated by the full subcategory C, and the inclusion $C \hookrightarrow D$ is a cΣa-morphism. Suppose D' is a continuous cΣa, and $F : C \longrightarrow D'$ is a cΣa- morphism. Then the extension $F^{\#} : D \longrightarrow D'$ of the Extension Lemma is a continuous cΣa-morphism.*

Proof. It only remains to show $F^{\#}$ is a cΣa-morphism. Again, the Extension Lemma is useful. Recall that D^n is compactly generated by C^n, for each nonnegative integer n. Thus, if $\sigma \in \Sigma_n$, there is an essentially unique extension to D^n of the functor

$$C^n \xrightarrow{\sigma_C} C \xrightarrow{F} D' \quad = \quad C^n \xrightarrow{F^n} (D')^n \xrightarrow{\sigma_{D'}} D'.$$

But both

$$D^n \xrightarrow{\sigma_D} D \xrightarrow{F^{\#}} D'$$

and

$$D^n \xrightarrow{(F^{\#})^n} (D')^n \xrightarrow{\sigma_{D'}} D'$$

are continuous extensions. The proof is complete. $\square$

We have proved the Characterization Theorem.

Theorem 5.1 (Characterization Theorem) *Suppose that C is a cΣa, and $\eta : C \longrightarrow D$ is a functor which is is full, faithful, injective on objects, and preserves initial objects. Then if D is compactly generated by $\eta(C)$, D admits the structure of a continuous cΣa in essentially one way such that (η, D) is the ω-completion of C.* □

Corollary 5.13 *If C is a sub-cΣa of the continuous cΣa D and D is compactly generated by C, then D is the ω-completion of C.* □

6 Existence of an ω-completion

In this section, we show that for any cΣa C, we may construct a cΣa C^ω and a morphism $\eta_\omega : C \longrightarrow C^\omega$ such that C^ω is compactly generated by $\eta_\omega(C)$. It follows that (η_ω, C^ω) is the completion of C. We will construct the cΣa C^ω as a quotient of the functor category $C^{\mathbb{N}}$.

6.1 Step 1.

We assume C has an initial object (if necessary, we adjoin one freely.)

Let $C^{\mathbb{N}}$ be the category whose objects are all functors $f : \mathbb{N} \longrightarrow C$; a morphism $\alpha : f \longrightarrow g$ is a natural transformation. We usually denote the components of a natural transformation $\alpha : f \longrightarrow g$ by

$$\alpha_n : f_n \longrightarrow g_n,$$

for $n \geq 0$.

Definition 6.1 (η_0 defined) *Let*

$$\eta_0 : C \longrightarrow C^{\mathbb{N}}$$

be the functor taking the object x in C to the functor $\eta_0(x)$ with $\eta_0(x)_n = x$, and $\eta_0(x)(n,p) = \mathbf{1}_x$, the identity morphism $x \longrightarrow x$, for all $0 \leq n \leq p$. On the morphism $g : x \longrightarrow y$ in C, the value of $\eta_0(g)$ is the natural transformation $\eta_0(x) \longrightarrow \eta_0(y)$, each of whose components is g.

Proposition 6.2 *The functor* $\eta_0 : C \longrightarrow C^{\mathbb{N}}$ *is full and faithful, and injective on objects. If* $\bot$ *is an initial object in* C, $\eta_0(\bot)$ *is initial in* $C^{\mathbb{N}}$. □

Now for the next step.

6.2 Step 2.

Definition 6.3 *Let* C^ω *be the category whose objects are those of* $C^{\mathbb{N}}$ *in which a morphism* $[\alpha] : f \longrightarrow g$ *is an* $\simeq$*-equivalence class of a weak map* $\alpha : f \longrightarrow mg$.

We define the canonical embedding of C into C^ω.

Definition 6.4 (η_ω defined) *Let* $\eta_\omega : C \longrightarrow C^\omega$ *be the functor taking* $f : x \longrightarrow y$ *in* C *to* $[\eta_0(f)] : \eta_0(x) \longrightarrow \eta_0(y)$ *in* C^ω.

We omit the proof of the following fact.

Proposition 6.5 *The functor* η_ω *is full, faithful, injective on objects, and preserves the initial object.* □

6.3 C^ω is compactly generated by $\eta_\omega(C)$

Here, we prove that C^ω is compactly generated by $\eta_\omega(C)$. Thus, when C^ω is equipped with the structure of a cΣa as in Proposition 5.11, (η_ω, C^ω) is the ω-completion of C.

Lemma 6.6 C^ω *has the completeness property and has finitary colimits. In detail, if* $f : \mathbb{N} \longrightarrow C$ *is any functor, then*

$$([\tau_n^f] : \eta_\omega(f_n) \quad \longrightarrow \quad f)_n$$

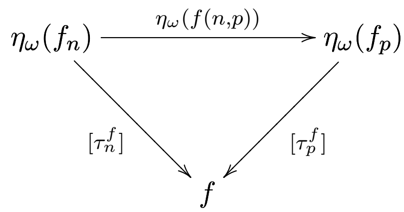
is a finitary colimiting cone, where $\eta_\omega(f_n) \longrightarrow \mu_n \cdot f$ *with components*

$$\tau_n^f(i) \quad := \quad f(n, \mu_n(i)). \tag{9}$$

(Recall, $\mu_n(i) = \max\{i, n\}$*.)*

Proof.

It is clear that for $n \leq p$, the diagram



commutes.

Now suppose that g is any object in C^ω, $([\nu^i] : \eta_\omega(f_i) \longrightarrow g)_i$ is a cocone over the diagram $f\eta_\omega$, where

$$\nu^i : \eta_\omega(f_i) \quad \longrightarrow \quad m_i g$$

is a weak map. Without loss of generality, we may assume

$$jm_i \quad \leq \quad jm_{i+1},$$

for all i, j.

But defining

$$\nu^\# : f \quad \longrightarrow \quad g$$

as the weak map with components

$$(\nu^\#)_i \quad := \quad \nu^i_i,$$

we have

$$\nu_i \quad \simeq \quad \tau_i^f \circ \nu^\#,$$

for each i, so that

$$[\nu_i] \quad = \quad [\tau_i^f] \cdot [\nu^\#].$$

Now suppose that c is an object in C and $\alpha : \eta_\omega(c) \longrightarrow m_\alpha \eta_\omega(f_n)$ and $\beta : \eta_\omega(c) \longrightarrow m_\beta \eta_\omega(f_n)$ are weak maps. Thus,

$$\begin{aligned} \alpha_i &= \alpha_0 \\ \beta_i &= \beta_0, \end{aligned}$$

for all $i \geq 0$. Suppose that $\alpha \circ \tau^n = \beta \circ \tau^n$. Then, for all i, there is some $j \geq im_\alpha, im_\beta$ such that

$$\begin{aligned}(\alpha \circ \tau^n)_i \cdot f(im_\alpha, j) &= \alpha_i \cdot f(n, \mu_n(im_\alpha)) \cdot f(im_\alpha, j)\\ &= (\beta \circ \tau^n)_i \cdot f(im_\beta, j)\\ &= \beta_i \cdot f(n, \mu_n(im_\beta)) \cdot f(im_\beta, j)\end{aligned}$$

Thus,

$$\alpha_i \cdot f(n, j) = \beta_i \cdot f(n, j).$$

But since $\alpha_i = \alpha_0$, and $\beta_i = \beta_0$, for all $i \geq 0$, this shows that

$$\alpha \cdot \eta_\omega(f(n, j)) = \alpha \cdot \eta_\omega(f(n, j)),$$

proving that the colimit $([\tau^n] : \eta_\omega(f_n) \longrightarrow f)_n$ is finitary. □

Corollary 6.7 (C^ω has finitary colimits) *For any functor $f : \mathbb{N} \longrightarrow C$, any colimiting cone of $f \cdot \eta_\omega$ in C^ω is finitary.*

We now consider the factorization property.

Lemma 6.8 (C^ω has the factorization property) *Suppose that c is an object in C, $f : \mathbb{N} \longrightarrow C$ is an object in C^ω, and $[\alpha] : c\eta_\omega \longrightarrow f$ is a morphism in C^ω. Then $[\alpha]$ factors as*

$$[\alpha] = [g\eta_\omega] \cdot [\tau^f_n],$$

for some $n \geq 0$, and some morphism $g : c \longrightarrow f_n$ in C.

Proof. If $\alpha : c\eta_\omega \longrightarrow mf$ is any weak map, then, for any i, since $(c\eta)(0, i) = \mathbf{1}_c$,

$$\alpha_i = c \xrightarrow{\alpha_0} f_{0m} \xrightarrow{f(0m,\ im)} f_{im}.$$

If $g = \alpha_0 : f_0 \longrightarrow f_{0m}$ in C, we have

$$[\alpha] = [g\eta_\omega] \cdot [\tau^f_{0m}]. \quad \square$$

Theorem 6.1 *C^ω is compactly generated by $\eta_\omega(C)$.*

Proof. By Corollary 6.7 and Lemma 6.8. □

We impose the structure of a cΣa using Proposition 5.11. One concrete description of the Σ-functors is this. Suppose, say, that $\sigma \in \Sigma_2$, and $f^i, g^i : \mathbb{N} \longrightarrow C$ are objects in C^ω, for $i = 1, 2$. Suppose also that $\alpha^i : f^i \longrightarrow g^i$ are weak maps. By the Inflation Lemma, we may assume that there is one endofunctor $m : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$\begin{aligned} \alpha^1 : f^1 &\longrightarrow mg^1 \\ \alpha^2 : f^2 &\longrightarrow mg^2. \end{aligned}$$

Then $\sigma_{C^\omega}(\alpha^1, \alpha^2)$ is the $\simeq$-equivalence class of the weak map

$$\sigma_C(f^1, f^2) \longrightarrow m\,\sigma_C(g^1, g^2)$$

with components

$$\sigma_C(\alpha^1_j, \alpha^2_j) : \sigma_C(f^1_j, f^2_j) \longrightarrow \sigma_C(g^1_{jm}, g^2_{jm}).$$

Here, $\sigma_C(f^1, f^2)$ is the functor

$$N \xrightarrow{\langle f^1,\ f^2\rangle} C \times C \xrightarrow{\sigma_C} C.$$

Similarly, for $\sigma_C(g^1, g^2)$.

Theorem 6.2 *(η_ω, C^ω) is the ω-completion of C.*

Proof. By Theorem 5.1. □

Proposition 6.9 *If s, t are Σ-terms in $\boldsymbol{Tm}_\Sigma(p)$, then $C \models s \preceq t$ iff $C^\omega \models s \preceq t$; similarly, $C \models s \cong t$ iff $C^\omega \models s \cong t$.* □

Proof. Suppose that $\lambda : s \longrightarrow t$ in C^2, say. Then there is a family of natural maps

$$\lambda_{x_1,x_2} : s(x_1, x_2) \longrightarrow t(x_1, x_2)$$

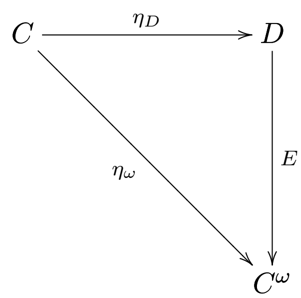
for each pair of C-objects x_1, x_2. These maps determine a weak map $\lambda : s(f_1, f_2) \longrightarrow t(f_1, f_2)$, for each pair of functors $f_1, f_2 : \mathbb{N} \longrightarrow C$ in the obvious way, and hence a natural transformation

$$[\lambda] : s(f_1, f_2) \longrightarrow t(f_1, f_2).$$

The converse direction follows from Proposition 6.5. The proof for $\cong$ is similar. □

Corollary 6.10 *Suppose that (η_D, D) is an ω-completion of C. Then η_D must be full, faithful, injective on objects, and preserves the initial object.*

Proof. There is an equivalence functor $E : D \longrightarrow C^\omega$ such that



and η_ω has these properties. It follows that η_D does also. □

Comparison

Johnstone [Ele02], in volume 2, C4.2, considers the "inductive" completion of a category, one in which for every directed (or "filtered") **category** I, each functor $I \longrightarrow C$ has a colimit. We have dealt only with the case I is the poset ω. His characterization of the Ind-completion of C is this:

- there is a full and faithful functor $y : C \longrightarrow \hat{C}$
- $\hat{C}$ has all filtered colimits, and each object in $\hat{C}$ is expressible as a filtered colimit of objects in $y(C)$.
- For each object c in C, the hom-functor $hom(c, -) : \hat{C} \longrightarrow \mathsf{Set}$ preserves filtered colimits.

Our factorization and finitary colimits condition are equivalent to Johnstone's third condition.

He also constructs $\hat{C}$ as the category whose objects are the functors $I \longrightarrow C$, where I is a small filtered category. A morphism from $D : I \longrightarrow C$ to $E : J \longrightarrow C$ is a family $(\varphi_i : i \in I)$, where each φ_i is an equivalence class of maps $f : D_i \longrightarrow E_j$, where $f : D_i \longrightarrow E_j$ is equivalent to $f' : D_i \longrightarrow E_k$ if there is some $t > j, k$ such that

$$f \cdot E(j,t) \quad = \quad f' \cdot E(k,t),$$

"subject to the obvious compatibility conditions".

Thus, it seems our completion theorem is not essentially new, except for the following considerations. We have considered categories enriched with an algebraic structure, and showed that the completion process also respects this structure. In addition, we have defined a notion of "inequation" and "equation" and proved that the completion of a cΣa C satisfies the same equations and inequations as C.

7 Conclusion

We have presented a completion theorem for categorical algebras that generalizes the well-known completion of ordered algebras from [Blo76]. We have shown that the completion C^ω is conservative in the sense that it satisfies all (in)equalities that hold in C. In addition to order completion, we have presented two main applications: synchronization trees and words, and thus found concrete descriptions of free continuous categorical algebras satisfying monoid and commutative monoid "equations". We believe that the Completion Theorem will find several more applications in Computer Science. For one example, the collection of countable labeled partial orders over an alphabet, sometimes called pomsets, equipped with the operations of series and parallel composition is a continuous categorical algebra in a natural way, cf. [Pra86, Ren96, LW00]. We expect that this algebra is equivalent to the completion of the categorical algebra determined by the finite pomsets. Further natural sources of applications are *event structures* (cf. [WN95]), or *labeled transition systems* with *bisimulations*, cf. [Mil89].

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