# Some Quasi-Varieties of Iteration Theories

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**Abstract**

All known structures involving a constructively obtainable fixed point (or iteration) operation satisfy the equational laws defining iteration theories. Hence, there seems to be a general equational theory of iteration. This paper provides evidence that there is no general *implicational* theory of iteration. In particular, the quasi-variety generated by the continuous ordered theories, in which fixed point equations have least solutions, is incomparable with the quasi-variety generated by the pointed iterative theories, in which fixed point equations have unique solutions.

This article appeared in the Proceedings of *Mathematical Foundations of Programming Semantics 1993*, Lecture Notes in Computer Science, volume 802, 378–409.

## 1 Introduction

Iteration theories were introduced in 1980 by Bloom, Elgot and Wright, and independently by Z. Ésik, in order to formalize the equational properties of the stepwise behavior of flowchart algorithms and to provide a calculus for solving systems of fixed point equations. Iteration theories, which are (Lawvere) algebraic theories enriched by a fixed point operation, have basic operations which, in the flowchart setting, denote composition, a case statement and a looping or iteration operation. It now appears that the equational laws of iteration theories are quite comprehensive. It has been shown that in all structures that have been used as semantic models, the equational properties of the fixed point operation are captured by the axioms describing iteration theories. These structures include

- the (equivalence classes) of the flowchart schemes themselves

*Partially supported by a joint grant from the NSF and the Hungarian Academy of Science
†Partially supported by a grant from the National Foundation for Scientific Research of Hungary and a joint grant from the NSF and the Hungarian Academy of Science

- $\omega$-continuous algebras
- theories of partial functions
- finitary and infinitary regular languages
- trees and synchronization trees
- the continuous functors involved in the specification of circular data types

and others.

Thus, the notion of iteration theory appears to be a unifying concept in many areas of theoretical computer science. We think it is important therefore to investigate various aspects of this notion. Equational axioms for iteration theories were given in [13, 14, 21, 22, 9, 16]. All of these sets of axioms involve a complicated equation scheme that we call the *commutative identity*. For example, in [13], other than the commutative identity, there are three equational schemes: the left and right zero identities and the pairing identity (see below).

Most of the known examples of iteration theories which are closely related to natural models of computation satisfy a simple implication scheme, the *functorial dagger implication*, which is much easier to establish than the commutative identity and which in fact implies the commutative identity. The quasi-variety *FD* of structures which are models of the functorial dagger implication, the zero identities, and the pairing identity, has the property that the least equational class containing *FD* is the class of all iteration theories. This fact is closely related to the fact, recently discovered independently by K.B. Arkhangelskii and P.V. Gorshkov [1], D. Kozen [18] and D. Krob [19] that the regular sets have simple finite implicational axiomatizations, although they have no finite equational axiomatization.

One might ask whether there is a general implicational theory of iteration, as general as the equational axioms determining the variety of iteration theories. In order to answer this question, we investigated the implicational theories of a number of quasi-varieties which are subclasses of the class of all iteration theories. Many of these quasi-varieties are of interest in themselves. Further, each has the property that the least variety it generates is either the variety of all iteration theories or the variety of all iteration theories with a unique morphism $1 \longrightarrow 0$. As is shown below, apparently there is no general implicational theory applicable to all of our examples. In particular, the quasi-variety $\Omega$ in which systems of fixed point equations have **least solutions**, and the quasi-variety *PI* in which (nontrivial) systems of fixed point equations have **unique solutions**, have incomparable implicational theories.

# 2 Preliminaries

In this section, we give the precise definitions needed to understand the later results. Familiarity with [7] or [8] would be helpful. We will use the following notation. For

$n \geq 0$, the set $[n]$ is

$$[n] = \{1, 2, \ldots, n\}.$$

In any category, the composite of morphisms $f : X \to Y$ and $g : Y \to Z$ is written $f \cdot g : X \to Z$.

We prefer the following definition of an algebraic theory.

**Definition 2.1** *An* **algebraic theory** *is a category $T$ whose objects are the nonnegative integers $n,\ n \geq 0$. For each $n \geq 0$, there are $n$* distinguished morphisms

$$i_n : 1 \to n$$

*with the following coproduct property. For any family of morphisms $f_i : 1 \to p$, for $i \in [n]$, there is a unique morphism $f : n \to p$ such that*

$$i_n \cdot f \quad = \quad f_i, \tag{1}$$

*for each $i \in [n]$. A* **morphism of algebraic theories** *$\varphi : T \longrightarrow T'$ is a functor which preserves objects and distinguished morphisms, i.e., $n\varphi = n$ and $i_n\varphi = i_n$, for all $n \geq 0$ and all $i \in [n]$.*

The morphism $f$ determined by (1) is called the *source tupling* of the morphisms $f_i$, and is written

$$f \quad = \quad \langle f_1, \ldots, f_n \rangle.$$

In the case that $n = 0$, the condition (1) amounts to the requirement that there is a unique morphism $0_p : 0 \to p$, for each $p$. When $n = 1$, we always assume that $f_1 = \langle f_1 \rangle$. For any $n \geq 0$, the identity morphism $n \to n$ will be denoted using boldface by $\mathbf{1}_n$. Note that $\mathbf{1}_n = \langle 1_n, 2_n, \ldots, n_n \rangle$.

Suppose that $T$ is an algebraic theory. For each (set theoretic) function $f : [n] \longrightarrow [p]$, there is a "base morphism" $f' : n \longrightarrow p$ defined as the source tupling of the distinguished morphisms $(if)_p : 1 \longrightarrow p,\ i \in [n]$. When $T$ is nontrivial, i.e., when there are at least 2 morphisms $1 \longrightarrow 2$ in $T$, the map from functions to base morphisms is injective. We will usually identify base morphisms $n \longrightarrow p$ with functions $[n] \longrightarrow [p]$. A base morphism is called surjective or a permutation, etc., when the corresponding function has that property.

The coproduct properties of theories imply that for any pair of morphisms $f : n \longrightarrow p$ and $g : m \longrightarrow p$ in $T$, there is a unique morphism $\langle f, g \rangle : n+m \longrightarrow p$ such that $\kappa \cdot \langle f, g \rangle = f$ and $\lambda \cdot \langle f, g \rangle = g$, where $\kappa : n \longrightarrow n+m$ and $\lambda : m \longrightarrow n+m$ are base morphisms corresponding to the inclusion and translated inclusion functions. The morphism $\langle f, g \rangle : n + m \longrightarrow p$ is called the **source pairing** of $f, g$.

For any pair of morphisms $f : n \longrightarrow p$ and $g : m \longrightarrow q$, we write $f \oplus g$ for the morphism $\langle f \cdot \kappa,\ g \cdot \lambda \rangle : n+m \longrightarrow p+q,$, where now $\kappa : p \longrightarrow p+q$ and $\lambda : q \longrightarrow p+q$. The morphism $f \oplus g$ is called the **separated sum** of $f, g$.

**Definition 2.2** *A* **preiteration theory** $T$ *is an algebraic theory enriched by an iteration operation* $f \mapsto f^\dagger$*, where* $f : n \longrightarrow n+p$ *and* $f^\dagger : n \longrightarrow p$*. A* **morphism of preiteration theories** $\varphi : T \longrightarrow T'$ *is a theory morphism which preserves the iteration operation, i.e.,* $f^\dagger\varphi = (f\varphi)^\dagger$*, for all* $f : n \longrightarrow n+p$ *in* $T$*.*

The operation $^\dagger$ need not satisfy any properties.

**Definition 2.3** *An* **iteration theory** *is a preiteration theory in which the iteration operation satisfies the following four identities: (see [13])*

- LEFT ZERO IDENTITY

  $$(0_n \oplus f)^\dagger \quad = \quad f,$$

  *all* $f : n \longrightarrow p$

- RIGHT ZERO IDENTITY

  $$(f \oplus 0_q)^\dagger \quad = \quad f^\dagger \oplus 0_q,$$

  *all* $f : n \longrightarrow n+p$*.*

- PAIRING IDENTITY

  $$\langle f, g\rangle^\dagger \quad = \quad \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle,\ h^\dagger\rangle$$

  *all* $f : n \longrightarrow n+m+p,\ g : m \longrightarrow n+m+p$*, where*

  $$h \quad := \quad g \cdot \langle f^\dagger,\ \mathbf{1}_{m+p}\rangle.$$

- COMMUTATIVE IDENTITY

  $$\langle 1_m \cdot \rho \cdot f \cdot (\rho_1 \oplus \mathbf{1}_p), \ldots, m_m \cdot \rho \cdot f \cdot (\rho_m \oplus \mathbf{1}_p)\rangle^\dagger \quad = \quad \rho \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger,$$

  *all* $f : n \longrightarrow m+p$*, surjective base* $\rho : m \longrightarrow n$*, and base* $\rho_i : m \longrightarrow m,\ i \in [m]$*, such that* $\rho_i \cdot \rho = \rho$*.*

The above four identities imply the following two:

- FIXED POINT IDENTITY

  $$f^\dagger \quad = \quad f \cdot \langle f^\dagger, \mathbf{1}_p\rangle,$$

  all $f : n \longrightarrow n+p$.

- PERMUTATION IDENTITY

  $$(\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p))^\dagger \quad = \quad \pi \cdot f^\dagger,$$

  for all $f : n \rightarrow n+p$ and all base permutations $\pi : n \rightarrow n$.

The commutative identity is the axiom which is most difficult to verify (and to understand!). By replacing it with certain implications, we will obtain some quasi-varieties which generate the class of all iteration theories.

First, we give a name to a simpler group of identities.

**Definition 2.4** *A* **Conway theory** *is a preiteration theory which satisfies the zero identities, the pairing identity and the permutation identity.*

The term Conway theory is due to the fact that in matrix preiteration theories, an equivalent set of identities is given by the familiar star sum and product identities which were studied by Conway [10]. See also [22, 5, 6].

$$\begin{aligned}(a+b)^* &= (a^*b)^*a^* \\ (ab)^* &= 1 + a(ba)^*b\end{aligned}$$

In any Conway theory we define $\perp := \mathbf{1}_1{}^\dagger : 1 \longrightarrow 0$ and $\perp_{np} := \langle \perp \cdot 0_p, \ldots, \perp \cdot 0_p \rangle$. It follows from the Conway theory axioms that

$$\perp_{np} \;=\; (\mathbf{1}_n \oplus 0_p)^\dagger,$$

for all $n, p \geq 0$.

Now we describe two implication schemes.

- Functorial dagger implication

  $$f \cdot (\rho \oplus \mathbf{1}_p) \;=\; \rho \cdot g \quad\Rightarrow\quad f^\dagger \;=\; \rho \cdot g^\dagger,$$

  for all $f : n \longrightarrow n + p$, $g : m \longrightarrow m + p$ and surjective base $\rho : n \longrightarrow m$. (When the implication holds for all morphisms $\rho$ in some class $\mathcal{C}$, we say that the theory satisfies the **functorial dagger for** $\mathcal{C}$.)

- The GA-gimplication

  $$f^{\dagger\dagger} \;=\; g^{\dagger\dagger} \quad\Rightarrow\quad (g \cdot \langle f^\dagger, \mathbf{1}_{1+p} \rangle)^\dagger \;=\; f^{\dagger\dagger},$$

  for all $f, g : 1 \longrightarrow 2 + p$.

Note that both the functorial dagger implication and the GA-implication in fact consist of infinitely many implications. The GA-implication was introduced in the setting of matrix theories in [1] to give a set of implicational axioms for the regular sets.

It is easy to verify the following fact.

**Proposition 2.5** [13] *If $T$ is a preiteration theory which satisfies the functorial dagger implication, then $T$ satisfies the commutative identity. Hence any Conway theory which satisfies the functorial dagger implication is an iteration theory.* □

The next proposition was proved in [8].

**Proposition 2.6** *If $T$ is a Conway theory which satisfies the GA-implication, then $T$ also satisfies the functorial dagger implication.* □

The first class of iteration theories we describe is a class of ordered theories.

**Definition 2.7**

- *An* **ordered algebraic theory** *$T$ is an algebraic theory such that for each pair $n, p$ of nonnegative integers, the set $T(n,p)$ is equipped with a partial order. The order on $T(n,p)$ will be written $f \leq g : n \to p$. The theory operations respect the ordering: if $f_1 \leq f_2 : n \to p$ and $g_1 \leq g_2 : p \to q$ then*

$$f_1 \cdot g_1 \;\leq\; f_2 \cdot g_2.$$

  *Further, if $f_i \leq g_i : 1 \to p$, for each $i \in [n]$, then*

$$\langle f_1, \ldots, f_n \rangle \;\leq\; \langle g_1, \ldots, g_n \rangle.$$

- *A* **pointed ordered theory** *is an ordered theory which is pointed; i.e., there is a distinguished morphism $\perp : 1 \longrightarrow 0$; as usual, we define $\perp_{1p}$ as $\perp \cdot 0_p$, for all $p \geq 0$, and $\perp_{np}$ as $\langle \perp_{1p}, \ldots, \perp_{1p} \rangle$, for $n \neq 1$. Furthermore, the morphisms $\perp_{np}$ are the* least *elements in $T(n,p)$. Note that composition in pointed theories is* **left strict***:*

$$\perp_{np} \cdot f \;=\; \perp_{nq},$$

  *for all $f : p \to q$.*

- *A pointed ordered theory $T$ is* $\omega$**-continuous** *if each hom-set $T(n,p)$ is an $\omega$-complete poset and if composition is also $\omega$-continuous:*

$$\begin{aligned} (\sup_n f_n) \cdot g &= \sup_n f_n \cdot g \\ f \cdot (\sup_n g_n) &= \sup_n f \cdot g_n, \end{aligned}$$

  *for $\omega$-chains $(f_n), (g_n)$, where $f_n : m \to p$ and $g_n : p \to q$, $n \geq 0$, and for $f : m \longrightarrow p$, $g : p \longrightarrow q$.*

(The importance of certain kinds of ordered theories for semantics, in particular the $\omega$-continuous theories, was emphasized by the ADJ group (J. Goguen, J. Thatcher, E.G. Wagner and J. Wright) in a number of papers [24, 17, 23].) In [8] it is shown that each $\omega$-continuous theory is an iteration theory, where for $f : n \longrightarrow n + p$,

$$f^\dagger \;:=\; \sup_k f^k \cdot \langle \perp_{np}, \mathbf{1}_p \rangle.$$

The powers $f^k$ of $f$ are defined as follows:

$$f^0 \;:=\; \mathbf{1}_n \oplus 0_p \tag{2}$$

$$f^{k+1} \;:=\; f \cdot \langle f^k, 0_n \oplus \mathbf{1}_p \rangle. \tag{3}$$

Thus, in $\omega$-continuous theories, $f^\dagger$ is the least solution to the iteration equation for $f$:

$$\xi \quad = \quad f \cdot \langle \xi, \mathbf{1}_p \rangle.$$

Now we recall a class of theories first introduced by Elgot [11].

**Definition 2.8** *An* **ideal theory** *is an algebraic theory $T$ with the property that each morphism $1 \longrightarrow p$ in $T$ is either a distinguished morphism $i_p$ or is* ideal. *An* **ideal morphism** *$f : 1 \longrightarrow p$ is a morphism with the property that for each $g : p \longrightarrow q$, the composite $f \cdot g : 1 \longrightarrow q$ is not distinguished. An* **iterative theory** *is an ideal theory such that for each ideal morphism $f : 1 \longrightarrow 1 + p$, there is a unique morphism $f^\dagger : 1 \longrightarrow p$ such that*

$$f^\dagger \quad = \quad f \cdot \langle f^\dagger, \mathbf{1}_p \rangle.$$

**Proposition 2.9** [3] *If $t_0 : 1 \longrightarrow 0$ is any morphism in an iterative theory $T$, there is a unique extension of the operation $^\dagger$ from the ideal morphisms to all morphisms such that $T$ becomes an iteration theory satisfying $\mathbf{1}_1{}^\dagger = t_0$. The resulting iteration theory is denoted* $(T,\, \mathbf{1}_1{}^\dagger = t_0)$. □

An iteration theory $(T,\, \mathbf{1}_1{}^\dagger = t_0)$, where $T$ is an iterative theory, is called a **pointed iterative theory**. In [8], it is shown that any pointed iterative theory satisfies the GA-implication.

Tree theories are examples of pointed iterative theories as well as $\omega$-continuous theories.

**Example 2.10** Theories of trees. Let $\Sigma$ be a *signature*, i.e., $\Sigma = \bigcup_{n \geq 0} \Sigma_n$ is the union of the pairwise disjoint sets $\Sigma_n$. Suppose the set $V = \{x_1, x_2, \ldots\}$ is disjoint from $\Sigma$. In the theory $\Sigma\,\mathbf{TR}$, a morphism $1 \to p$ is a $\Sigma$-tree $t : 1 \to p$. (A $\Sigma$-tree $t : 1 \longrightarrow p$ is a partial function whose domain is a nonempty prefix closed subset of the set $[\omega]^*$ of finite sequences of positive integers. The target of $t$ is the set $\Sigma \cup \{x_1, \ldots, x_p\}$. Further, if $u \in \operatorname{dom} t$ and $u\,t \in \Sigma_n$ then $ui \in \operatorname{dom} t$ iff $i \in [n]$; also, if $u\,t \in \Sigma_0 \cup \{x_1, \ldots, x_p\}$, then $u$ is a leaf; i.e., $ui$ is not in $\operatorname{dom} t$, for any $i > 0$. See [12] for a thorough study of the algebraic theory of trees.) We identify the varible $x_i$ with the partial function defined only on the empty word $\lambda$ with value $x_i$. Similarly, we identify $\sigma \in \Sigma_n$ with the partial function defined on $\lambda$ and the length one sequences $1, \ldots, n$ as follows: $\lambda\,\sigma := \sigma$; $i\,\sigma := x_i$, $i \in [n]$.

If $n \neq 1$, a morphism $n \to p$ in $\Sigma\,\mathbf{TR}$ is an $n$-tuple $(t_1, \ldots, t_n)$ of morphisms $1 \to p$. The **composite** of $t : 1 \to p$ with $s = (s_1, \ldots, s_p) : p \to q$ is the tree obtained by attaching the tree $s_i$ to each leaf of $t$ labeled $x_i$, $i \in [p]$. When $n \neq 1$, the composite of $t = (t_1, \ldots, t_n) : n \to p$ with $s : p \to q$ is defined as

$$t \cdot s \quad := \quad (t_1 \cdot s, \ldots, t_n \cdot s).$$

The distinguished morphism $i_n$ is the tree $x_i : 1 \to n$, for each $i \in [n]$.

Note that if $f : 1 \longrightarrow 1 + p$ is any tree other than $1_{1+p}$, there is a unique tree $f^\dagger : 1 \longrightarrow p$ such that

$$f^\dagger \quad = \quad f \cdot \langle f^\dagger, \mathbf{1}_p \rangle.$$

Thus, $\Sigma\,\mathbf{TR}$ is an iterative theory.

If $\perp$ is a letter not in the set $\Sigma$, let $\Sigma_\perp$ denote the signature obtained by adding $\perp$ to $\Sigma_0$. The pointed iterative theory

$$(\Sigma_\perp\,\mathbf{TR},\ \mathbf{1}_1{}^\dagger = \perp)$$

is an $\omega$-continuous ordered theory, where the ordering can be described as follows: $f \leq g$ if $g$ can be obtained from $f$ by replacing some occurrences of $\perp$ by other trees. In fact, $(\Sigma_\perp\,\mathbf{TR},\ \mathbf{1}_1{}^\dagger = \perp)$ is the free $\omega$-continuous theory on $\Sigma$ (see [24, 17]).

A **tree theory** is a pointed iterative theory

$$(\Sigma_\perp\,\mathbf{TR},\ \mathbf{1}_1{}^\dagger = \perp),$$

where the "point" $\perp$ is the new atomic letter of rank 0. This theory is usually written $\Sigma_\perp\,\mathbf{TR}$, for short.

We will occasionally make use of the subiteration theory $\Sigma\mathbf{tr}$ of $\Sigma_\perp\,\mathbf{TR}$, which consists of the **regular** trees. Recall that a tree $t : 1 \longrightarrow p$ is regular if it has a finite number of subtrees. When $t : n \longrightarrow p$, for some $n \neq 1$, $t$ is regular if the components $i_n \cdot t$ are regular, for all $i \in [n]$. According to the definition in [3, 4] that $\Sigma\mathbf{tr}$ can be characterized as the iteration theory freely generated by $\Sigma$.

We mention two other important classes of $\omega$-continuous theories.

**Example 2.11** The theory $\mathbf{Rel}_A$ has as morphisms $n \longrightarrow p$ all relations

$$A \times [n] \quad\longrightarrow\quad A \times [p].$$

Composition in the theory is composition of relations. Identifying $A$ with $A \times [1]$, the distinguished morphism $i_n : 1 \to n$, is the function, considered as a relation,

$$\begin{array}{ccc} A & \to & A \times [n] \\ a & \mapsto & (a, i). \end{array}$$

The tupling $f := \langle f_1, \ldots, f_n\rangle$ of the morphisms $f_i : 1 \to p$ is the relation

$$\begin{array}{rcl} A \times [n] & \to & A \times [p] \\ (a,i)\ f\ (a',j) & \Leftrightarrow & a\ f_i\ (a',j). \end{array}$$

With the standard ordering of relations, for any relation $f : A \times [n] \longrightarrow A \times [n+p]$, there is a least relation $f^\dagger : A \times [n] \longrightarrow A \times [p]$ which satisfies

$$f^\dagger \quad=\quad f \cdot \langle f^\dagger, \mathbf{1}_p\rangle.$$

The theory $\mathbf{Rel}_A$ is also an $\omega$-continuous theory. The morphism $\perp_{np}$ in $\mathbf{Rel}_A$ is the empty relation $n \longrightarrow p$.

The theory $\mathbf{Pfn}_A$ is the subtheory of $\mathbf{Rel}_A$ whose morphisms $n \longrightarrow p$ are the partial functions $A \times [n] \longrightarrow A \times [p]$. The distinguished morphisms and the theory and iteration operations are the same as those in $\mathbf{Rel}_A$.

The notion of an *iteration term* is defined in the expected way as a formal expression which denotes a morphism in a (pre)iteration theory. These terms are constructed from an infinite set of variables for morphisms $n \longrightarrow p$, for each $n, p \geq 0$, constants for each of the distinguished morphisms, and the operation symbols for composition, tupling and dagger. (Source pairing and separated sum are understood as abbreviations.)

For our purposes, an *implication* or *quasi-identity* is an expression of the form

$$s_1 = t_1 \wedge \ldots \wedge s_n = t_n \quad \Rightarrow \quad s_{n+1} = t_{n+1},$$

where $n \geq 0$ and $s_i, t_i$ are iteration terms. Note that when $n = 0$, an implication is an equation. Each instance of the functorial dagger or GA-implication is an implication. We understand an implication to be **satisfied** by, or **true** or **valid** in a preiteration theory $T$ if the implication is true for any interpretation of the variables as morphisms in $T$ with the appropriate source and target.

If $K$ is any class of preiteration theories, let $Imp(K)$ denote the collection of all implications valid in each theory in $K$. For any set $I$ of implications, let $Mod(I)$ denote all preiteration theories in which each implication in $I$ is true.

Suppose that $K$ is a class of iteration theories.

**Definition 2.12** *The* **quasi-variety generated by** $K$*, in symbols* $Qv(K)$*, is the class* $Mod(Imp(K))$*, the class of all iteration theories which satisfy all implications valid in all theories in* $K$*.* $K$ *is a* **quasi-variety** *if* $K = Qv(K)$*.*

Note that if $K \subseteq K'$, then $Qv(K) \subseteq Qv(K')$. If $I$ is some set of implications, then the class of all models of $I$ is a quasi-variety.

# 3 Some Quasi-Varieties of Iteration Theories

Aside from *IT*, the variety of all iteration theories, we will be considering the following quasi-varieties.

- $V_t$, the quasi-variety generated by the class of tree theories.
- $V_g$, the quasi-variety generated by the class of theories

$$(\Sigma\, \mathbf{TR},\ \mathbf{1}_1{}^\dagger = t_0).$$

- *PI*, the quasi-variety generated by all pointed iterative theories.
- *PFN*, the quasi-variety generated by the theories $\mathbf{Pfn}_A$.
- *REL*, the quasi-variety generated by the theories $\mathbf{Rel}_A$.
- $\Omega$, the quasi-variety generated by the class of all $\omega$-continuous theories.
- $\Omega_0$, the quasi-variety generated by all $\omega$-continuous theories with a unique morphism $1 \longrightarrow 0$.

- *MAT*, the quasi-variety generated by all matrix iteration theories [8, 5].
- *GA*, the collection of Conway theories which satisfy the GA-implication.
- *FD*, the collection of Conway theories which satisfy the functorial dagger implication.
- Manes [20] has called a morphism $h : n \longrightarrow p$ in a preiteration theory *pure* if $h \cdot \mathbf{1}_p{}^\dagger = \mathbf{1}_n{}^\dagger$. Note that any base morphism is pure. Let $FD_{\mathrm{p}}$ denote the collection of Conway theories which satisfy the functorial dagger implication for the class of all pure morphisms, namely, the implication

$$h \cdot \mathbf{1}_p{}^\dagger = \mathbf{1}_n{}^\dagger \ \wedge \ f \cdot (h \oplus \mathbf{1}_p) = h \cdot g \quad \Rightarrow \quad f^\dagger = h \cdot g^\dagger,$$

  for all $f : n \longrightarrow n + p$, $g : m \longrightarrow m + p$, and $h : n \longrightarrow m$.
- Lastly, we let $FD_{\mathrm{s}}$ denote the quasi-variety of all Conway theories which satisfy the functorial dagger implication for the class of all morphisms.

# 4 The Results

We will prove that if the quasi-varieties are ordered by set inclusion, they form a poset whose structure is indicated in Figure 1.

Each of the inclusions is strict. If there is no chain of inclusions from one quasi-variety to another, the two are incomparable with respect to inclusion. Thus, we have a complete description of the poset of these classes.

Further, the variety of iteration theories generated by $V_t$, and all of the quasi-varieties above it, is *IT*. The variety generated by *PFN*, *REL*, $\Omega_0$, $FD_{\mathrm{s}}$, and *MAT* is the variety of all iteration theories with a unique morphism $1 \longrightarrow 0$.

In addition, we will show that there is an infinite chain of quasi-varieties

$$FD \subset \ldots \subset FD_3 \subset FD_2 \subset FD_1 = IT$$

between *FD* and *IT*. $FD_n$ is the quasi-variety of all iteration theories which satisfy the functorial dagger implication for the class of base morphisms $k \longrightarrow 1$, for $1 \leq k \leq n$.

# 5 Inclusions

It is clear that the following inclusions hold:

$$\begin{array}{lllllll} V_t & \subseteq & V_g & \subseteq & PI & & \\ FD_{\mathrm{s}} & \subseteq & FD_{\mathrm{p}} & \subseteq & FD & & \\ PFN & \subseteq & REL & \subseteq & \Omega_0 & \subseteq & \Omega. \end{array}$$

and each class is contained in the class *IT* of all iteration theories. The inclusions

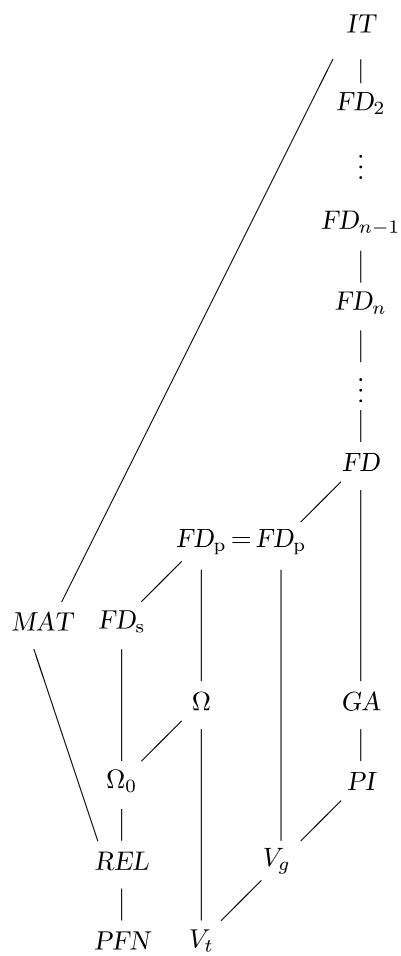


Figure 1: The Quasi-Variety Poset

$$\begin{array}{lllll} PI & \subseteq & GA & \subseteq & FD \\ \Omega_0 & \subseteq & FD_{\mathrm{s}} & & \\ REL & \subseteq & MAT & & \end{array}$$

are proved in [8], and the inclusion

$$V_t \quad \subseteq \quad \Omega$$

is known from [24]. We will show now that each of the inclusions indicated in Figure 1, page 11, is proper, and that two quasi-varieties are incomparable unless there is a chain of inclusions from one to the other. It was proved in [15] that $FD \neq IT$. We will give two new independent arguments for this fact below.

The organization of the argument follows the shape of Figure 1. We proceed first up the right side, and continue to the left.

### 5.1 $V_t \subset V_g$

We need show only that $V_t \neq V_g$. Consider the following implication.

$$f \cdot g = \mathbf{1}_1{}^\dagger \quad \Rightarrow \quad f^\dagger = \mathbf{1}_1{}^\dagger, \quad \text{all } f : 1 \longrightarrow 1,\ g : 1 \longrightarrow 0. \tag{4}$$

This implication is valid in all tree theories, since in the tree theory

$$(\Sigma_\perp \, \mathbf{TR},\, \mathbf{1}_1{}^\dagger = \perp),$$

the tree $\mathbf{1}_1{}^\dagger$ is the atomic tree $\perp$. Thus, if $f \cdot g = \perp$, then either $f = \mathbf{1}_1$ (and $g = \perp$) or $f = \perp \cdot 0_1$. However, if $\Sigma$ contains a letter $f \in \Sigma_1$ and a letter $g$ in $\Sigma_0$, the implication fails in the theory

$$(\Sigma_\perp \, \mathbf{TR},\, \mathbf{1}_1{}^\dagger = f \cdot g).$$

### 5.2 $V_g \subset PI$

The inclusion $V_g \subseteq PI$ holds since every theory $(\Sigma \, \mathbf{TR},\, \mathbf{1}_1{}^\dagger = t_0)$ is a pointed iterative theory. Also the following implication is valid in these theories:

$$f \cdot f = g \cdot g \quad \Rightarrow \quad f = g, \quad f, g : 1 \longrightarrow 1. \tag{5}$$

But we show that (5) is not valid in an iterative theory $\mathbf{TTF}_X$ of timed terminal functions on $X$ [11]. Indeed, writing $\mathbf{N}$ for the set of nonnegative integers, let $X = \{a, b\}$ and let $f, g : X \times \mathbf{N} \longrightarrow X \times \mathbf{N}$ be the timed terminal functions defined by

$$\begin{array}{lll} (a,n)f & := & (b, n+1) \\ (b,n)f & := & (a, n+1) \\ (x,n)g & := & (x, n+1), \quad x \in \{a, b\}. \end{array}$$

Then $(x,n)f^2 = (x,n)g^2 = (x, n+2)$, but $f \neq g$.

### 5.3 $PI \subset GA$

It was shown in [8] that $PI \subseteq GA$. In order to show the inclusion is strict, we will show that the following implication is valid in *PI*.

$$p \cdot q = \mathbf{1}_1 \quad \Rightarrow \quad \langle p,\ p \cdot \pi \rangle \cdot \langle p,\ p \cdot \pi \rangle = \mathbf{1}_2, \tag{6}$$

where $p : 1 \longrightarrow 2,\ q : 2 \longrightarrow 1$ and where $\pi := \langle 0_1 \oplus \mathbf{1}_1, \mathbf{1}_1 \oplus 0_1 \rangle$ is the nontrivial base permutation $2 \longrightarrow 2$. Indeed, in any ideal theory, if $p : 1 \longrightarrow 2$ and if $p \cdot q = \mathbf{1}_1$ for some $q : 2 \longrightarrow 1$, then either $p = \mathbf{1}_1 \oplus 0_1$ or $p = 0_1 \oplus \mathbf{1}_1$. Hence $f := \langle p,\ p \cdot \pi \rangle$ is either $\mathbf{1}_2$ or $\pi$. In either case $f \cdot f = \mathbf{1}_2$.

If $S$ is any semiring, $\mathbf{Mat}_S$ is the theory whose morphisms $n \longrightarrow p$ are the $n$ by $p$ matrices with entries in $S$; matrix multiplication is the theory composition. For other details, see [5, 8]. When $S$ is the semiring of regular subsets of $A^*$, we denote the corresponding matrix theory by $\mathbf{Reg}_A$. Now suppose that $T = \mathbf{Reg}_A$, and that $p$ and $q$ are the following matrices:

$$\begin{aligned} p &:= [1\ x] \\ q &:= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \end{aligned}$$

where $x$ is some nonempty regular subset of $A^*$. Then $p \cdot q = \mathbf{1}_1$. However if $f = \langle p,\ p \cdot \pi \rangle$, then

$$\begin{aligned} f &= \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \\ f \cdot f &= \begin{bmatrix} 1 + x^2 & x \\ x & 1 + x^2 \end{bmatrix} \\ &\neq \mathbf{1}_2. \end{aligned}$$

It is known from [1] that $\mathbf{Reg}_A$ satisfies the GA-implication. Hence, $\mathbf{Reg}_A$ is in $GA - PI$.

**Remark 5.1** We note here that the collection of all pointed iterative theories does not form a quasi-variety since this collection is not closed under binary products. For the same reason, the collection of tree theories or theories $(\Sigma_\perp \mathbf{TR},\ {\mathbf{1}_1}^\dagger = t_0)$ is not a quasi-variety.

### 5.4 $GA \subset FD$

It was shown in [8] that $GA \subseteq FD$. In order to show that the inclusion is strict, we apply the following extension of the Zero Congruence Lemma [8].

**Lemma 5.2** *If $\theta$ is a zero congruence on the free iteration theory $\Sigma\mathbf{tr}$, then the theory $\Sigma\mathbf{tr}/\theta$ satisfies the functorial dagger implication.*

We give a proof of this fact, together with a concrete description of $\theta$, in the Appendix.

Now define $\Sigma$ as the signature having only two symbols $f, g$ of rank 2. Let $\theta$ be the zero congruence generated by

$$f^{\dagger\dagger} \quad \theta \quad g^{\dagger\dagger}.$$

In the Appendix (Theorem 8.2) it is shown that two regular trees $1 \longrightarrow p$ are related by $\theta$ if one can be obtained from the other by replacing some subtrees of the form $f^{\dagger\dagger}$ by $g^{\dagger\dagger}$, and some subtrees $g^{\dagger\dagger}$ by $f^{\dagger\dagger}$. The theory $\Sigma\mathbf{tr}/\theta$ satisfies the functorial dagger implication, by the lemma, but does not satisfy the GA-implication. Indeed, if

$$h \quad := \quad g \cdot \langle f^\dagger, \mathbf{1}_1 \rangle,$$

then it is not the case that $h^\dagger \ \theta \ f^{\dagger\dagger}$, since $h^\dagger$ has no subtrees of the form $f^{\dagger\dagger}$ or $g^{\dagger\dagger}$.

## 5.5 $V_g \subset FD_{\mathrm{p}}$

First, we prove the following lemma.

**Lemma 5.3** *Suppose that*

$$T := (\Sigma\,\mathbf{TR},\ \mathbf{1}_1{}^\dagger = t_0).$$

*Then* $T \in FD_{\mathrm{p}}$.

We write $\perp$ for $\mathbf{1}_1{}^\dagger$, as usual, rather than $t_0$. In [4] it was shown that for any $f : n \longrightarrow n+p$ in $T$, $f^\dagger$ is given by a metric limit:

$$f^\dagger \quad = \quad \lim_{k \longrightarrow \infty} f^k \cdot (\perp_{n0} \oplus \mathbf{1}_p),$$

where $f^k$ was defined in (2) and (3) above. Now, suppose that $h : n \longrightarrow m$ and that

$$f \cdot (h \oplus \mathbf{1}_p) \quad = \quad h \cdot g,$$

where $g : m \longrightarrow m + p$. Then it follows that for each $k \geq 1$,

$$f^k \cdot (h \oplus \mathbf{1}_p) \quad = \quad h \cdot g^k.$$

Hence, if $h$ is pure,

$$\begin{aligned} f^k \cdot (\perp_{n0} \oplus \mathbf{1}_p) &= f^k \cdot (h \cdot \perp_{m0} \oplus \mathbf{1}_p) \\ &= f^k \cdot (h \oplus \mathbf{1}_p) \cdot (\perp_{m0} \oplus \mathbf{1}_p) \\ &= h \cdot g^k \cdot (\perp_{m0} \oplus \mathbf{1}_p). \end{aligned}$$

The result follows from the fact that

$$\begin{aligned} \lim_k h \cdot g^k \cdot (\perp_{m0} \oplus \mathbf{1}_p) &= h \cdot (\lim_k g^k \cdot (\perp_{m0} \oplus \mathbf{1}_p)) \\ &= h \cdot g^\dagger. \quad \square \end{aligned}$$

**Corollary 5.4** $V_g \subseteq FD_{\mathrm{p}}$. $\square$

To show that the inclusion $V_g \subseteq FD_{\mathrm{p}}$ is proper, note that the theory $\mathbf{TTF}_X$ of timed terminal functions in section 5.2 is in *PFN* as well as in *PI*, and hence in $FD_{\mathrm{p}}$. It was shown in that section that $\mathbf{TTF}_X$ is not in $V_g$.

## 5.6 $FD_{\mathrm{p}} \subset FD$

We now find an iteration theory which satisfies the functorial dagger implication for all base morphisms but not for all pure morphisms. Let $\Sigma$ be a signature which has just two symbols $\sigma, \tau$ in $\Sigma_1$, and which is empty otherwise. We define an iteration theory congruence $\sim$ on $\Sigma_\perp\,\mathbf{TR}$ as follows. For any trees $f, g : 1 \longrightarrow p$, $f \sim g$ iff

- both trees have a leaf labeled by some variable $x_i$, $i \in [p]$, and the set of all labels of all of the vertices of $f$ are the same as those of $g$, or
- neither has leaf labeled by a variable, and both symbols $\sigma, \tau$ occur **infinitely often** in both $f$ and $g$, or
- neither tree has a leaf labeled by a variable, and neither tree has both symbols $\sigma, \tau$ occurring infinitely often as vertex labels.

Of course, if $f, g : n \longrightarrow p$, where $n > 1$, then $f \sim g$ iff $i_n \cdot f \sim i_n \cdot g$, for all $i \in [n]$. Let $T$ denote the quotient theory $\Sigma_\perp\,\mathbf{TR}/\!\sim$. Then the morphisms $1 \longrightarrow 0$ in $T$ are the two congruence classes

$$[\perp] \quad \text{and} \quad [(\sigma \cdot \tau)^\dagger].$$

When $p \geq 1$, the morphisms $1 \longrightarrow p$ in $T$ are the $4p+2$ equivalence classes

$$[\perp \cdot 0_p],\ [(\sigma \cdot \tau)^\dagger \cdot 0_p],\ [i_p],\ [\sigma \cdot i_p],\ [\tau \cdot i_p],\ [\sigma \cdot \tau \cdot i_p]$$

for $i \in [p]$. It is easy to check that $T$ satisfies the functorial dagger implication for any base $\rho$ with target 1. It follows from [16], that $T$ satisfies the functorial dagger implication for all base morphisms. Since

$$[\sigma] \cdot [\perp] \quad = \quad [\perp],$$

$[\sigma] : 1 \longrightarrow 1$ is pure. Also,

$$[\sigma \cdot \tau] \cdot [\sigma] \quad = \quad [\sigma] \cdot [\tau].$$

But $[\tau^\dagger] = [\perp]$, so that

$$\begin{aligned} [(\sigma \cdot \tau)^\dagger] \quad &\neq \quad [\perp] \\ &= \quad [\sigma] \cdot [\tau^\dagger]. \end{aligned}$$

Hence $T$ is in $FD - FD_{\mathrm{p}}$.

## 5.7 $V_t \subset \Omega$

Again, we need show only that $V_t \neq \Omega$. Consider the implication

$$f^\dagger = \mathbf{1}_1{}^\dagger \quad \Rightarrow \quad f \cdot f = f, \quad \text{all } f : 1 \longrightarrow 1. \tag{7}$$

This implication is clearly valid in all tree theories, since in $\Sigma_\perp\,\mathbf{TR}$, if $f : 1 \longrightarrow 1$ and $f^\dagger = \perp$, then either $f = \mathbf{1}_1$ or $f = \perp \cdot 0_1$.

We show that if $A$ is a set with at least 2 elements, the implication does not hold in $\mathbf{Pfn}_A$. Indeed, let $f$ be a nontrivial permutation of $A$ of order two. Then, since there is a unique morphism $1 \longrightarrow 0$ in $\mathbf{Pfn}_A$, $f^\dagger = \mathbf{1}_1{}^\dagger$. But $f \cdot f \neq f$.

## 5.8 $\Omega \subset FD_{\mathrm{p}}$

Using an argument just like that given above for Lemma 5.3, replacing $\lim_k$ with $\sup_k$, it can be shown that $\Omega \subseteq FD_{\mathrm{p}}$.

Now we show that the two quasi-varieties are distinct. The implication

$$f \cdot g = \bot \quad \Rightarrow \quad f^\dagger = \bot, \quad \text{all } f : 1 \longrightarrow 1,\ g : 1 \longrightarrow 0, \tag{8}$$

holds in $\Omega$. Indeed, in any $\omega$-continuous theory, if $f \cdot g = \bot$,

$$\begin{aligned} \bot &\leq f \cdot \bot \\ &\leq f \cdot g \\ &= \bot. \end{aligned}$$

Hence $f \cdot \bot = \bot$, which in turn implies $f^n \cdot \bot = \bot$, all $n \geq 1$. Thus $f^\dagger = \sup_n f^n \cdot \bot = \bot$.

However, the implication (8) fails in the theory

$$T \quad := \quad (\Sigma\, \mathbf{TR},\, \mathbf{1}_1{}^\dagger = t_0)$$

when $\Sigma_1$ contains the letter $\sigma$ say, and $\Sigma_0$ contains the letter $\delta$ and $t_0 := \sigma \cdot \delta$. But $T \in FD_{\mathrm{p}}$, by Corollary 5.4. Thus $T \in FD_{\mathrm{p}} - \Omega$.

## 5.9 $\Omega_0 \subset \Omega$

This follows immediately from the fact that there are theories in $\Omega$ with more than one morphism $1 \longrightarrow 0$, e.g. tree theories.

## 5.10 $FD_{\mathrm{s}} \subset FD_{\mathrm{p}}$

Since it is clear that $FD_{\mathrm{s}} \subseteq FD_{\mathrm{p}}$, we show that there is a theory in $FD_{\mathrm{p}} - FD_{\mathrm{s}}$. Indeed, choose any theory $T$ in $V_g$ with at least two morphisms $1 \longrightarrow 0$. Then $T$ cannot be in $FD_{\mathrm{s}}$, since any such theory has a unique morphism $1 \longrightarrow 0$ (for a proof, see [8], for example.)

## 5.11 $PFN \subset REL$

Since each partial function $A \times [n] \longrightarrow A \times [p]$ is a relation, $PFN \subseteq REL$. We show the inclusion is strict.

For any morphism $p : 1 \longrightarrow 2$ in a preiteration theory, define the two morphisms $p_T, p_F : 1 \longrightarrow 1$ as follows:

$$\begin{aligned} p_T &:= p \cdot (\mathbf{1}_1 \oplus \bot) \\ p_F &:= p \cdot (\bot \oplus \mathbf{1}_1). \end{aligned}$$

The following implication is valid in *PFN*:

$$p \cdot \langle \mathbf{1}_1, \mathbf{1}_1 \rangle = \mathbf{1}_1 \quad \Rightarrow \quad p_T \cdot p_F = \bot_{1,1}. \tag{9}$$

Indeed, if $p : A \longrightarrow A \times [2]$ is a partial function, then if $p \cdot \langle \mathbf{1}_1, \mathbf{1}_1 \rangle = \mathbf{1}_1$, then for each $a \in A$, either $ap = (a, 1)$ or $ap = (a, 2)$. If $ap = (a, 1)$ then $ap_F$ is undefined, $ap_T$ is defined, and $ap_T = a$. If $ap = (a, 2)$, then $ap_T$ is undefined and $ap_F = a$. In either case, $ap_T \cdot p_F$ is undefined. Thus, $p_T \cdot p_F = \bot_{1,1}$.

To see that this implication (9) is not valid in *REL*, let $A := \{x\}$ and suppose that $p : 1 \longrightarrow 2$ is the relation satisfying $xp = \{(x, 1), (x, 2)\}$. Then $p \cdot \langle \mathbf{1}_1, \mathbf{1}_1 \rangle = \mathbf{1}_1 = p_T \cdot p_F$.

## 5.12 $REL \subset \Omega_0$

Note that the following implication is valid in *REL*.

$$\begin{aligned} & p \cdot \langle \mathbf{1}_1, \mathbf{1}_1 \rangle = \mathbf{1}_1 \ \wedge \ p_T \cdot p_F = \bot_{1,1} \\ & \quad \Rightarrow \quad p \cdot (p \oplus p) = p \cdot (\mathbf{1}_1 \oplus 0_2 \oplus \mathbf{1}_1), \end{aligned} \tag{10}$$

where $p : 1 \longrightarrow 2$. The morphisms $p_T, p_F : 1 \longrightarrow 1$ are defined above in section 5.11. Indeed, if the equation $p \cdot \langle \mathbf{1}_1, \mathbf{1}_1 \rangle = \mathbf{1}_1$ holds in $\mathbf{Rel}_A$, then for each $a \in A$, $a\ p\ (a, 1)$ or $a\ p\ (a, 2)$, or both. The second equation guarantees that at most one of these conditions can hold, so that in fact, $p$ is a total function $A \longrightarrow A \times [2]$ which takes $a \in A$ to either $(a, 1)$ or $(a, 2)$. It follows that if the hypotheses of (10) hold in $\mathbf{Rel}_A$, so does the conclusion.

But let $D$ be the three element chain $\text{BOT} < a < \text{TOP}$. Consider the least subtheory $T$ of $\mathbf{Pow}_D$, the theory of all functions on $D$, containing the constant function BOT and the meet function $x \wedge y$. A morphism $f : 1 \longrightarrow p$ in $T$ is either the constant function $D^p \longrightarrow D$ with value BOT or $f(x_1, \ldots, x_p) = \bigwedge\{x_i : i \in I\}$, for some nonempty $I \subseteq [p]$. Thus, the unique morphism $\bot : 1 \longrightarrow 0$ in $T$ has the value BOT. It follows that $T$ is a subiteration theory of the theory of all continuous functions $D^p \longrightarrow D^n$. Hence $T$ is in $\Omega$, and

$$f^\dagger \quad := \quad \bigvee_n f^n \cdot (\bot \oplus \mathbf{1}_p),$$

for $f : 1 \longrightarrow 1 + p$. But $T$ does not satisfy the implication (10). Indeed, let $p(x_1, x_2) = x_1 \wedge x_2$. Then

$$\begin{aligned} p(x, x) &= x \\ p_T \cdot p_F(x) &= x \wedge \text{BOT} \\ &= \bot_{1,1}. \end{aligned}$$

However,

$$\begin{aligned} p \cdot (p \oplus p) &= p(p(x, y), p(, u, v)) \\ &= x \wedge y \wedge u \wedge v \\ &\neq p \cdot (\mathbf{1}_1 \oplus 0_2 \oplus \mathbf{1}_1) \\ &= p(x, v) \ = \ x \wedge v. \end{aligned}$$

## 5.13 $\Omega_0 \subset FD_{\mathrm{s}}$

In each theory in $\Omega_0$, every morphism is pure. Since $\Omega \subseteq FD_{\mathrm{p}}$, it follows that $\Omega_0 \subseteq FD_{\mathrm{s}}$.

We will use the following implication, which holds in $\Omega_0$:

$$f \cdot \langle \bot_{1,1}, \mathbf{1}_1 \rangle = \mathbf{1}_1 \wedge f \cdot \langle \mathbf{1}_1, \mathbf{1}_1 \rangle = \mathbf{1}_1 \quad \Rightarrow \quad f^\dagger = \mathbf{1}_1,$$

where $f : 1 \longrightarrow 2$. Indeed, note that if the hypotheses of the implication hold for $f$, then $f^n \cdot \langle \bot_{1,1,}, \mathbf{1}_1 \rangle = \mathbf{1}_1$, for all $n \geq 1$.

Now consider the three element idempotent star semiring $S_0 = \{0, 1, 1^*\}$, with the star defined by

$$x^* \quad := \quad \begin{cases} 1 & x = 0 \\ 1^* & \text{otherwise.} \end{cases}$$

This semiring is $\omega$-complete, when any infinite sum with infinitely many nonzero summands is defined to be $1^*$. It follows that the matrix theory $\mathbf{Mat}_{S_0}$ is an iteration theory (see [8]). The iteration operation applied to any $n$ by $n+p$ matrix $f = [a \quad b]$ (where $a$ is $n$ by $n$ and $b$ is $n$ by $p$), yields

$$f^\dagger \quad := \quad a^* b,$$

where $a^* = \sum_{n=0}^{\infty} a^n$.

Now if $T = \mathbf{Mat}_S$ and if $S$ is *any* $\omega$-complete semiring, then $T$ is in $FD_{\mathrm{s}}$ [5, 8]. But in $\mathbf{Mat}_{S0}$, let $f := [1 \quad 1]$. Then

$$\begin{aligned} f \cdot \langle \bot_{1,1}, \mathbf{1}_1 \rangle &= [1 \quad 1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= f \cdot \langle \mathbf{1}_1, \mathbf{1}_1 \rangle \\ &= [1 \quad 1] \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= 1 \end{aligned}$$

but $f^\dagger = 1^*$. Hence $T \in FD_{\mathrm{s}} - \Omega_0$.

## 5.14 $REL \subset MAT$

It is known from [8] that $REL \subseteq MAT$. The theory $T = \mathbf{Mat}_{S_0}$ in the previous section does not belong to $\Omega_0$, and hence does not belong to $REL$.

## 5.15 $MAT \subset IT$

This follows immediately from the fact that all theories in $MAT$ have a unique morphism $1 \longrightarrow 0$.

# 6 Incomparable Quasi-Varieties

It is obvious that $V_t - FD_s$ and $V_t - MAT$ are nonempty, since all theories in $FD_s$ and $MAT$ have a unique morphism $1 \longrightarrow 0$. Thus, by inspecting Figure 1, page 11, it will be seen that all of the incomparability results will follow once we can prove that each of the following classes is nonempty:

$$PFN - GA, \quad PI - FD_{\mathrm{p}}, \quad FD_{\mathrm{s}} - \Omega, \quad \Omega_0 - MAT, \quad MAT - FD_2. \tag{11}$$

Indeed, for example, if $PFN \subseteq X$ and $Y \subseteq GA$, for some quasi-varieties $X, Y$, then it follows from the fact that $PFN - GA$ is nonempty that $X - Y \neq \emptyset$.

We now proceed to prove the statements in (11).

## 6.1 $PFN - GA \neq \emptyset$

Let $A := \{a, b\}$ and let $f, g : A \longrightarrow A \times [3]$ be the following (total) functions.

$$\begin{aligned} af &:= (b, 1) \\ bf &:= (b, 3) \\ ag &:= (b, 3) \\ bg &:= (a, 2) \end{aligned}$$

Then

$$af^{\dagger\dagger} = bf^{\dagger\dagger} = ag^{\dagger\dagger} = bg^{\dagger\dagger} = (b, 1).$$

Hence $f^{\dagger\dagger} = g^{\dagger\dagger}$ is the total function $A \longrightarrow A$ with value $b$. Now if the GA-implication were true in $\mathbf{Pfn}_A$,

$$f^{\dagger\dagger} = h^{\dagger},$$

where $h := f \cdot \langle g^{\dagger}, \mathbf{1}_2 \rangle$. But note that

$$\begin{aligned} ah &= bg^{\dagger} \\ &= (a, 1), \end{aligned}$$

so that $h^{\dagger}$ is not defined on $a$.

## 6.2 $PI - FD_{\mathrm{p}} \neq \emptyset$

Let $\Sigma$ consist of two letters $\{g, h\}$ of rank 1, and let $T$ be the quotient theory of the free iteration theory $\Sigma\mathbf{tr}$ with respect to the smallest iteration theory congruence $\theta$ such that

$$g \cdot h \equiv g \pmod{\theta}, \quad g \cdot \perp \equiv \perp \pmod{\theta}, \quad g^{\dagger} \equiv \perp \pmod{\theta}.$$

The morphisms $1 \longrightarrow 1$ in $T$ are the congruence classes of the following trees:

- $h^p \cdot g^q \cdot h^\dagger \cdot 0_1, \quad q \neq 0$
- $h^\dagger \cdot 0_1$
- $h^p \cdot \perp \cdot 0_1$
- $h^p \cdot g^q$.

The theory $T$ is ideal, and for any ideal $r : 1 \longrightarrow 1$, the fixed point equation $x = r \cdot x$ has a unique solution. Indeed, consider the equation $x = h^n \cdot g^m \cdot x$, where $x : 1 \longrightarrow 0$. If $m \neq 0$, the unique solution is $h^n \cdot \perp$. When $m = 0$ but $n \neq 0$, the unique solution is $h^\dagger$. Thus $T$ is in *PI*.

But $T$ is not in $FD_{\mathrm{p}}$. Indeed,

$$\mathbf{1}_1 \cdot g \;\equiv\; g \cdot h \pmod{\theta},$$

and $g$ is pure, but

$$\mathbf{1}_1{}^\dagger \;=\; \perp \;\not\equiv\; g \cdot h^\dagger \pmod{\theta}.$$

## 6.3 $FD_{\mathrm{s}} - \Omega \neq \emptyset$

This fact follows immediately from the fact that $FD_{\mathrm{s}} - \Omega_0 \neq \emptyset$, proved above in Section 5.13.

## 6.4 $\Omega_0 - MAT \neq \emptyset$

The example in Section 5.13 shows that there is some theory in $MAT - \Omega_0$. We now show that there is a theory in $\Omega_0 - MAT$.

Let $T$ be the least subiteration theory of the iteration theory of all order preserving functions on the 3 element chain $\textsc{bot} < a < \textsc{top}$ containing the constant function $\textsc{bot}$ and the function

$$f(x,y) \;:=\; \begin{cases} y & \text{if } x = \textsc{bot} \\ \textsc{bot} & \text{if } y = \textsc{bot} \\ \textsc{top} & \text{otherwise.} \end{cases}$$

We note that

$$\begin{aligned} f^\dagger(x) &= f(x,x) \\ &= \begin{cases} \textsc{bot} & \text{if } x = \textsc{bot} \\ \textsc{top} & \text{otherwise} \end{cases} \\ f^{\dagger\dagger} &= \textsc{bot}. \end{aligned}$$

Since $T$ has a unique morphism $1 \longrightarrow 0$, $T$ is in $\Omega_0$.

The following implication is valid in $MAT$. For all $f, g : 1 \longrightarrow 2$, if

$$\begin{array}{rcll} f \cdot \langle \mathbf{1}_1, \perp_{1,1} \rangle & = & g \cdot \langle \mathbf{1}_1, \perp_{1,1} \rangle & \text{and} \\ f \cdot \langle \perp_{1,1}, \mathbf{1}_1 \rangle & = & g \cdot \langle \perp_{1,1}, \mathbf{1}_1 \rangle & \end{array}$$

then

$$f \quad = \quad g.$$

Indeed, write $f = [a, b],\ g = [c, d]$. Then the first equation says $a = c$ and the second says $b = d$.

Now in the above theory $T$, let $g = 0_1 \oplus \mathbf{1}_1$; i.e., $g(x, y) := y$. Then

$$\begin{array}{rcl} f \cdot \langle \perp_{1,1}, \mathbf{1}_1 \rangle & = & f(\textsc{bot}, x) \\ & = & x \\ & = & g(\textsc{bot}, x); \\ f \cdot \langle \mathbf{1}_1, \perp_{1,1} \rangle & = & f(x, \textsc{bot}) \\ & = & \textsc{bot} \\ & = & g(x, \textsc{bot}). \end{array}$$

But $f \neq g$, since, e.g., $f(a, a) = \textsc{top}$ and $g(a, a) = a$.

## 6.5 $MAT - FD_2 \neq \emptyset$

We will construct a matrix theory $T$ over a semiring which is a quotient of the semiring of regular subsets of $\Sigma^*$, where $\Sigma = \{a, b, c, d\}$. The congruence is the least $^*$-semiring congruence which identifies the sets $\{a, b\}$ and $\{c, d\}$.

Recall, that for any language $L \subseteq \Sigma^*$, the set of **factors** of $L$ is defined as follows.

$$fac\, L \quad := \quad \{v \in \Sigma^* : uvw \in L,\ \text{some } u, v \in \Sigma^*\}.$$

An $n$**-state nondeterministic finite automaton** (nfa) over $\Sigma$ is a triple $M = (\alpha, A, \gamma)$, where $A$ is an $n \times n$ matrix whose entries are subsets of $\Sigma$, and where $\alpha : 1 \longrightarrow n$ and $\beta : n \longrightarrow 1$ are 0-1 matrices. The **behavior** of $(\alpha, A, \gamma)$ is the regular language

$$|M| \quad := \quad \alpha A^* \gamma.$$

The **states** of $M$ are the integers in $[n]$; there is an edge $(i, j)$ with source $i$ and target $j$ in $M$ if $A_{i,j}$ is nonempty, in which case a label of such an edge is any letter in the set $A_{i,j}$. (Equivalently, one might say there is one edge from $i$ to $j$ for each letter in $A_{i,j}$.) For states $j, j'$, a **path** from $j$ to $j'$ is a sequence of states $j = i_0, i_1, \ldots, i_m = j'$, such that $(i_k, i_{k+1})$ is an edge, for each $k < m$; a **label** of such a path is any word $x_1 \ldots x_m \in \Sigma^m$ such that $x_k \in A_{i_{k-1}, i_k}$, for $0 < k \leq m$. The **initial states** of $M$ are those states $i$ such that $\alpha_{1,i} = 1$; the **final states** are those states $j$ such that $\gamma_{j,1} = 1$. The **accessible states** are those on paths whose source is an initial state; the **coaccessible states** are those on paths whose target is a final state.

It may easily be shown that the behavior of $(\alpha, A, \gamma)$ is the set of all words which label paths from an initial state to a final state, since $A^* = \bigcup_{k\geq 0} A^k$.

Thus, if $L$ is the behavior of some nfa $M$, $v \in \mathit{fac}\, L$ iff $v$ is a label of some path in $M$ whose source is an accessible state and whose target is a coaccessible state.

For a positive integer $K$, say that a language is **$K$-bounded** if for all words $u$, and all nonnegative integers $m$,

$$(u(a+b))^m \subseteq \mathit{fac}\, L \quad \Rightarrow \quad m < K, \quad \text{and} \tag{12}$$

$$(u(c+d))^m \subseteq \mathit{fac}\, L \quad \Rightarrow \quad m < K. \tag{13}$$

$L$ is **bounded** if $L$ is $K$-bounded, for some integer $K$. Note that for any $L, u, n, m$, if

$$(u(c+d))^{n+m} \subseteq \mathit{fac}\, L$$

then

$$(u(c+d))^n \subseteq \mathit{fac}\, L.$$

**Example 6.1** The language

$$(a+b+c+d)^*$$

is not bounded, and

$$(a+bd^*c)^*(1+bd^*)$$

is 2-bounded. □

Let $M = (\alpha, A, \gamma)$ and $M' = (\alpha, B, \gamma)$ be two $n$-state finite automata, with the same initial and final states. We write $M \sim M'$ if there is an edge $(i,j)$ of $M$ with $A_{i,j} = \{a,b\} \cup Z$ and $B_{i,j} = \{c,d\}$ or if $A_{i,j} = \{c,d\} \cup Z$ and $B_{i,j} = \{a,b\}$. (Here, in the first case $Z$ is some subset of $\{c,d\}$ and in the second, $Z$ is a subset of $\{a,b\}$.) Thus, $M \sim M'$ if it is possible to obtain $M'$ by changing the set of labels of one edge of $M$ by replacing $\{a,b\}$ by $\{c,d\}$ or vice versa. Note that the set of accessible or coaccessible states in $M$ are exactly those in $M'$.

**Theorem 6.2** *Suppose that $M = (\alpha, A, \gamma)$ and $M' = (\alpha, B, \gamma)$ are automata with $M \sim M'$. If $L = |M|$ is $K$-bounded, then $R = |M'|$ is $2K$-bounded.*

*Proof.* Let $e = (i,j)$ be the edge in $M$ whose label is changed in order to obtain $M'$. Suppose that $L$ is $K$-bounded, and in order to produce a contradiction, suppose that the implication

$$\forall u \forall m\, [(u(c+d))^m \subseteq \mathit{fac}\, R \quad \Rightarrow \quad m < 2K].$$

is false. Then there is a word $u$ and some integer $m \geq 2K$ such that

$$(u(c+d))^m \quad \subseteq \quad \mathit{fac}\, R.$$

As noted above, it then follows that

$$\begin{aligned} (u(c+d))^{2K} \quad &\subseteq \quad \mathit{fac}\, R \quad \text{and} \\ (u(c+d))^{K} \quad &\subseteq \quad \mathit{fac}\, R. \end{aligned}$$

Since $L$ is $K$-bounded, there is some word $w$ in $(u(c+d))^K$ with the property that there is no path labeled $w$ in $M$ whose source is an accessible state and whose target is coaccessible. But there is such a path in $M'$, so this path must use the edge $e$ whose label was changed. Similarly, there is a path in $M'$ whose source is accessible, whose target is coaccessible, and whose label is $ww$, since $ww \in (u(c+d))^{2K}$. Thus, this second path must use the edge $e$ twice, showing that the edge $e$ lies on a cycle all of whose vertices are both accessible and coaccessible. If $v : j \longrightarrow i$ is a label of the rest of the cycle, then $v$ is a label of a path $j \longrightarrow i$ in both $M$ and $M'$, and if $a, b \in A_{i,j}$, then $(v(a+b))^m \subseteq \mathit{fac}\, L$, for all $m$, contradicting the hypothesis. (Similarly if $c, d \in A_{i,j}$, then $(v(c+d))^m \subseteq \mathit{fac}\, L$, all $m$.)

The same argument shows that the implication

$$\forall u\, \forall m\, [(u(a+b))^m \subseteq \mathit{fac}\, R \quad \Rightarrow \quad m < 2K]$$

holds as well. Thus, $R$ is $2K$-bounded. □

For languages $L, L'$, say $L \sim L'$ if there are automata $M, M'$ with $M \sim M'$, where $L$ is the behavior of $M$ and $L'$ is the behavior of $M'$.

**Definition 6.3** *For languages $L, R$, say $L \approx R$ if either $L = R$ or there is a finite sequence $L_1, L_2, \ldots, L_n$ of languages such that $L = L_1$, $R = L_n$ and $L_i \sim L_{i+1}$, for $i < n$.*

We omit the proof of the following theorem, which makes use of some of the constructions in Chapter 9, Section 4 of [8].

**Theorem 6.4** *The relation $\approx$ is the least congruence relation on the $^*$-semiring of regular subsets of $\Sigma^*$ such that $\{a, b\} \approx \{c, d\}$*

**Corollary 6.5** *Suppose that $L \approx R$ and $L$ is bounded. Then $R$ is bounded.*

*Proof.* This follows immediately from Theorem 6.2. □

Now let $T = \mathbf{Mat}_S$, where $S$ is the quotient of the regular subsets of $\Sigma^*$ by $\approx$. Define $A : 2 \longrightarrow 2$ as the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then, if $\rho : 2 \longrightarrow 1$ is the base morphism, we have

$$A \cdot \rho \quad = \quad \rho \cdot [\{a, b, c, d\}]$$

in $T$, since $\{a, b\} \approx \{c, d\} \approx \{a, b, c, d\}$. If $T$ satisfies the functorial dagger implication for the base morphism with source 2, then

$$A^* \cdot \rho \quad = \quad \rho \cdot \{a, b, c, d\}^*.$$

But

$$A^* \;=\; \left[ \begin{array}{ll} (a+bd^*c)^* & (a+bd^*c)^*bd^* \\ (d+ca^*b)^*ca^* & (d+ca^*b)^* \end{array} \right].$$

Letting $L$ denote the sum of the first two entries of $A^*$, we see that $L$ is bounded. But $\{a,b,c,d\}^*$ is not (see Example 6.1). Hence $A^* \cdot \rho \neq \rho \cdot \{a,b,c,d\}^*$ in $T$. Thus $T \in MAT - FD_2$. □

Note that $T$ is an example of an iteration theory not satisfying the functorial dagger implication.

# 7 A Chain of Quasi-Varieties

In this section we prove that there is an infinite number of quasi-varieties of iteration theories between *FD* and *IT*. It was shown in [13] that if a Conway theory $T$ has a functorial dagger for the base morphism $n \longrightarrow 1$, where $n \geq 1$ is any integer, then $T$ has a functorial dagger for all (surjective) base morphisms $n \longrightarrow m$. The following proposition can be proved in straightforward way.

**Proposition 7.1** *If a Conway theory has a functorial dagger for the base morphism $n \longrightarrow 1$, where $n \geq 1$ is a given integer, then it has a functorial dagger for all base morphisms $m \longrightarrow 1$, $m \in [n]$.* □

Let $n \geq 1$ be an integer. Recall that $FD_n$ denotes the quasi-variety of iteration theories satisfying the quasi-identity

$$f \cdot (\rho \oplus \mathbf{1}_p) \;=\; \rho \cdot g \quad \Rightarrow \quad f^\dagger \;=\; \rho \cdot g^\dagger, \tag{14}$$

where $f : n \longrightarrow n{+}p$, $g : 1 \longrightarrow 1{+}p$, and where $\rho$ denotes the base morphism $n \longrightarrow 1$. Thus $FD_1 = IT$ and $FD$ is the intersection of the quasi-varieties $FD_n$. By Proposition 7.1, we have

$$FD_n \;\subseteq\; FD_{n-1},$$

for all $n \geq 2$. Below we show that each inclusion is proper.

We assume that $n \geq 2$ is a fixed integer. We will be considering trees in $\Sigma_\perp \mathbf{TR}$, where $\Sigma_n = \{\sigma_1, \ldots, \sigma_n\}$, and $\Sigma_k = \emptyset$, for $k \neq n$. We let $X_p := \{x_1, \ldots, x_p\}$ be the set of the first $p$ variables.

**Definition 7.2** *Suppose $f : 1 \longrightarrow p$. We say that $f$ is* **perfect** *if $f \neq \perp \cdot 0_p$ and the following conditions hold for all $u \in [n]^*$.*

- *If $ui \in \mathrm{dom}\, f$, for some $i \in [n]$, then $f(ui) = \sigma_i$ or $f(ui) \in X_p$.*
- *If $f(u1), \ldots, f(un)$ are all in $X_p$, then not all values are the same variable.*

*A tree $n \longrightarrow p$ is perfect if each $i_n \cdot f$ is perfect, for all $i \in [n]$.*

Thus there are $n$ perfect trees $1 \longrightarrow 0$.

**Definition 7.3** *Suppose that $f, g : 1 \longrightarrow p$. We define $f \sim g$ iff $f = g$ or neither $f$ nor $g$ is perfect. When $f, g : n \longrightarrow p$, for some $n \neq 1$, then $f \sim g$ iff $i_n \cdot f \sim i_n \cdot g$, for all $i \in [n]$.*

Thus all non-perfect scalar trees $1 \longrightarrow p$ are identified. We list some elementary consequences of the definition.

1. If a tree $t : 1 \longrightarrow p$ has a non-perfect subtree, then $t$ is not perfect.
2. If $f$ is not perfect then $f \cdot g$ is not perfect.
3. If $f$ is not perfect then $f^\dagger$ is not perfect.
4. If $f$ is perfect, $x_i$ occurs in $f$, and if the $i$-th component of $g$ is not perfect or has root labeled $\sigma_j$ for some $j \neq i$, then $f \cdot g$ is not perfect.

**Proposition 7.4** *The relation $\sim$ is an iteration theory congruence on trees.*

*Proof.* This follows from the above facts. □

Note that each perfect tree $1 \longrightarrow p$ forms a singleton $\sim$-congruence class. For the rest of this section we let $T := \Sigma_\perp \mathbf{TR}/\sim$.

**Proposition 7.5** *The theory $T$ is contained in $FD_{n-1} - FD_n$.*

*Proof.* Suppose that

$$f \cdot (\rho \oplus \mathbf{1}_p) \sim \rho \cdot g,$$

where $f : k \longrightarrow k + p$, $g : 1 \longrightarrow 1 + p$ are trees, and where $\rho$ is the base morphism $k \longrightarrow 1$. Suppose that $1 \leq k < n$. We will show that $f^\dagger \sim \rho \cdot g^\dagger$.

*Case 1*: $g$ is perfect. Then $f$ is perfect and

$$f \cdot (\rho \oplus \mathbf{1}_p) \;=\; \rho \cdot g.$$

Thus,

$$f^\dagger \;=\; \rho \cdot g^\dagger,$$

since $\Sigma_\perp \mathbf{TR}$ has a functorial dagger.

*Case 2*: $g$ is not perfect. Then for each $i \in [k]$, either $i_k \cdot f$ is not perfect or $i_k \cdot f$ is perfect but has a subtree of the form

$$\sigma(z_1, \ldots, z_n),$$

where $\sigma \in \Sigma$ and the $z_j$'s are variables in $X_k$. If $i_k \cdot f$ is not perfect, then

$$i_k \cdot f^\dagger \;=\; i_k \cdot f \cdot \langle f^\dagger, \mathbf{1}_p \rangle$$

is not perfect. If $i_k \cdot f$ has a subtree of the form $\sigma(z_1, \ldots, z_n)$, then, since $k < n$, at least two of the $z_j$'s must be the same. Suppose that $z_1 = z_2$, say. But then

$$i_k \cdot f^\dagger \;=\; i_k \cdot f \cdot \langle f^\dagger, \mathbf{1}_p \rangle \;=\; i_k \cdot f \cdot \langle f \cdot \langle f^\dagger, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle$$

has a subtree of the form

$$\sigma(t, t, t_3, \ldots, t_n),$$

where none of the trees $t, t_3, \ldots, t_n$ is a variable. Again, it follows that $i_k \cdot f^\dagger$ is not perfect.

Thus $T$ is in $FD_{n-1}$. To prove that $T$ is not in $FD_n$, consider the tree

$$f \;:=\; \langle \sigma_1, \ldots, \sigma_n \rangle : n \longrightarrow n.$$

If $\rho$ denotes the base morphism $n \longrightarrow 1$ then

$$f \cdot \rho \sim \rho \cdot \sigma_1 \cdot \rho,$$

but

$$1_n \cdot f^\dagger \not\sim (\sigma_1 \cdot \rho)^\dagger,$$

since the tree $1_n \cdot f^\dagger$ is perfect and $(\sigma_1 \cdot \rho)^\dagger$ is not. □

Any theory in $FD_{n-1} - FD_n$ is another example of an iteration theory which does not satisfy the functorial dagger implication. Thus, we have infinitely many examples of such theories.

# 8 Appendix

Suppose that $T$ is any theory. A **zero congruence** $\Theta$ on $T$ is a theory congruence which is generated by an equivalence relation $\theta_0$ on the morphisms with target 0; i.e., $\Theta$ is the smallest congruence on $T$ such that $\theta_0 \subseteq \Theta$. Zero congruences were first considered in [2], where it was shown that the smallest zero congruence on any iteration theory is in fact an iteration theory congruence. In [8], this result was extended to show that in any preiteration theory satisfying the parameter identity, namely

$$(f \cdot (\mathbf{1}_n \oplus g))^\dagger \quad = \quad f^\dagger \cdot g, \quad \text{all } f : n \longrightarrow n + p,\ g : p \longrightarrow q,$$

any zero congruence preserves the dagger operation.

We prove here the following theorem.

**Theorem 8.1** *Suppose that $\Theta$ is a zero congruence on a free iteration theory $\Sigma\mathbf{tr}$ generated by an equivalence relation $\theta_0$ on the morphisms with target 0. Then for any $f, g : n \longrightarrow p$ in $\Sigma\mathbf{tr}$, $f \;\Theta\; g$ iff for some $F : n \longrightarrow p + k$, and some $\alpha, \beta : k \longrightarrow 0$,*

$$\begin{aligned} f &= F \cdot (\mathbf{1}_p \oplus \alpha) \\ g &= F \cdot (\mathbf{1}_p \oplus \beta) \quad \textit{and} \\ \alpha &\equiv \beta \pmod{\theta_0} \end{aligned}$$

**Corollary 8.2** *With the notation of Theorem 8.1, let $T$ denote the quotient theory $\Sigma\mathbf{tr}/\Theta$. Then the functorial dagger implication holds in $T$.*

The following proposition, the **Zero Congruence Lemma**, is known from [2]. See also [8] for further refinements.

**Proposition 8.3** *For any theory $T$, the least theory congruence $\Theta$ containing $\theta_0$ is the transitive closure of the following relation $\sim$: for $f, g : n \longrightarrow p$, $f \sim g$ iff for some $F : n \longrightarrow p + k$, and some $\alpha, \beta : k \longrightarrow 0$,*

$$\begin{aligned} f &= F \cdot (\mathbf{1}_p \oplus \alpha) \\ g &= F \cdot (\mathbf{1}_p \oplus \beta) \quad \textit{and} \\ \alpha &\equiv \beta \pmod{\theta_0}. \quad \square \end{aligned}$$

We will show that in $\Sigma\mathbf{tr}$, the relation $f \sim g$ just defined coincides with the relation $\Theta$.

We make use of the following lemma, proved in the next section.

**Lemma 8.4** *Suppose that $F \cdot (\mathbf{1}_p \oplus \beta) = G \cdot (\mathbf{1}_p \oplus \gamma)$ for some $F : n \longrightarrow p + k$, $G : n \longrightarrow p + k'$, $\beta : k \longrightarrow 0$, $\gamma : k' \longrightarrow 0$ in $\Sigma\mathbf{tr}$. Then for some integer $m \geq 0$, there are trees $H : n \longrightarrow p + m$ and $Q : m \longrightarrow k$, $Q' : m \longrightarrow k'$ in $\Sigma\mathbf{tr}$ such that*

$$\begin{aligned} F &= H \cdot (\mathbf{1}_p \oplus Q) \\ G &= H \cdot (\mathbf{1}_p \oplus Q') \\ Q \cdot \beta &= Q' \cdot \gamma. \end{aligned}$$

**Corollary 8.5** *The relations $\sim$ of Proposition 8.3 and $\Theta$ are the same.*

*Proof.* We need show only that $\sim$ is transitive, since $\Theta$ is the transitive closure of $\sim$. Assume that $f \sim g$, and $g \sim h$, where $f, g, h : n \longrightarrow p$ in $\Sigma\mathbf{tr}$. Then

$$\begin{aligned} f &= F \cdot (\mathbf{1}_p \oplus \alpha) \\ g &= F \cdot (\mathbf{1}_p \oplus \beta), \end{aligned}$$

for some $F : n \longrightarrow p + k$, some $\alpha, \beta : k \longrightarrow 0$ with $\alpha \; \theta_0 \; \beta$. Also,

$$\begin{aligned} g &= G \cdot (\mathbf{1}_p \oplus \gamma) \\ h &= G \cdot (\mathbf{1}_p \oplus \delta), \end{aligned}$$

for some $G : n \longrightarrow p + k'$, some $\gamma, \delta : k' \longrightarrow 0$ with $\gamma \; \theta_0 \; \delta$.

Since $g = F \cdot (\mathbf{1}_p \oplus \beta) = G \cdot (\mathbf{1}_p \oplus \gamma)$, then, by the lemma, for some trees $H, Q, Q'$ in $\Sigma\mathbf{tr}$, $F = H \cdot (\mathbf{1}_p \oplus Q)$, $G = H \cdot (\mathbf{1}_p \oplus Q')$ and $Q \cdot \beta = Q' \cdot \gamma$. But then

$$\begin{aligned} f &= H \cdot (\mathbf{1}_p \oplus Q) \cdot (\mathbf{1}_p \oplus \alpha) \\ &= H \cdot (\mathbf{1}_p \oplus (Q \cdot \alpha)) \\ h &= H \cdot (\mathbf{1}_p \oplus Q') \cdot (\mathbf{1}_p \oplus \delta) \\ &= H \cdot (\mathbf{1}_p \oplus (Q' \cdot \delta)) \end{aligned}$$

Also,

$$Q \cdot \alpha \equiv Q \cdot \beta \;=\; Q' \cdot \gamma \equiv Q' \cdot \delta \pmod{\theta_0}.$$

Thus, $Q \cdot \alpha \equiv Q' \cdot \delta \pmod{\theta_0}$, showing $f \sim h$. □

*Proof of Corollary 8.2.* It is enough to prove that the implication

$$f \cdot (\rho \oplus \mathbf{1}_p) \sim \rho \cdot g \quad \Rightarrow \quad f^\dagger \sim \rho \cdot g^\dagger$$

holds, when $f : n \longrightarrow n + p$, $g : 1 \longrightarrow 1 + p$ in $\Sigma\mathbf{tr}$ and when $\rho : n \longrightarrow 1$ is the unique base morphism. Thus, suppose that

$$f \cdot (\rho \oplus \mathbf{1}_p) \quad \sim \quad \rho \cdot g.$$

By definition, then, there is some $F : n \longrightarrow 1 + p + k$, some $\alpha, \beta : k \longrightarrow 0$ in $\Sigma\mathbf{tr}$ with

$$\begin{aligned} f \cdot (\rho \oplus \mathbf{1}_p) &= F \cdot (\mathbf{1}_{1+p} \oplus \alpha) \\ \rho \cdot g &= F \cdot (\mathbf{1}_{1+p} \oplus \beta), \quad \text{and} \\ \alpha &\equiv \beta \pmod{\theta_0}. \end{aligned}$$

Write $F = \langle F_1, \ldots, F_n \rangle$. Since $\rho \cdot g = F \cdot (\mathbf{1}_{1+p} \oplus \beta)$, it follows that

$$g \quad = \quad F_i \cdot (\mathbf{1}_{1+p} \oplus \beta),$$

for each $i \in [n]$. We will define the tree $G_i : 1 \longrightarrow n + p + k$ for each $i \in [n]$, such that if

$$G \quad := \quad \langle G_1, \ldots, G_n \rangle : n \longrightarrow n + p + k$$

then

$$\begin{aligned} f &= G \cdot (\mathbf{1}_{n+p} \oplus \alpha) \\ F &= G \cdot (\rho \oplus \mathbf{1}_{p+k}). \end{aligned}$$

Indeed, for each $i \in [n]$, let $G_i$ be obtained from $F_i$ by relabeling any leaf vertex $u$ of $F_i$ labeled $x_1$ by $u(i_n \cdot f)$ and by relabeling any leaf $v$ of $F_i$ labeled $x_{1+j}$, $j \in [p + k]$, by $x_{n+j}$. (Necessarily $u(i_n \cdot f) = x_j$, for some $j \in [n]$.)

Note the following facts.

$$\begin{aligned} f \cdot (\rho \oplus \mathbf{1}_p) &= G \cdot (\rho \oplus \mathbf{1}_p \oplus \alpha) \\ \rho \cdot g &= G \cdot (\rho \oplus \mathbf{1}_p \oplus \beta). \end{aligned}$$

We will prove that

$$G^\dagger \cdot (\mathbf{1}_p \oplus \beta) \quad = \quad \rho \cdot g^\dagger. \tag{15}$$

Indeed,

$$G \cdot (\mathbf{1}_{n+p} \oplus \beta) \cdot (\rho \oplus \mathbf{1}_p) \quad = \quad \rho \cdot g,$$

and since the functorial dagger implication is valid in $\Sigma\mathbf{tr}$, it follows that (15) holds. But then, by the parameter identity,

$$\begin{aligned} f^\dagger &= G^\dagger \cdot (\mathbf{1}_p \oplus \alpha) \\ &\sim G^\dagger \cdot (\mathbf{1}_p \oplus \beta) \\ &= \rho \cdot g^\dagger. \quad \square \end{aligned}$$

One application of these results was given in Section 5.4 above. Assume that $f, g \in \Sigma_2$, and let $\Theta$ be the zero congruence on $\Sigma\mathbf{tr}$ generated by the least equivalence such that $f^{\dagger\dagger} \sim g^{\dagger\dagger}$. Since both $f^{\dagger\dagger}$ and $g^{\dagger\dagger}$ are morphisms $1 \longrightarrow 0$, the theory $\Sigma\mathbf{tr}/\Theta$ satisfies the functorial dagger implication, by Corollary 8.2. Note that the tree $f^{\dagger\dagger}$ is the complete binary tree with each vertex labeled $f$. If $h : 1 \longrightarrow 1$ is the tree

$$h \quad := \quad g \cdot \langle f^\dagger, \mathbf{1}_1 \rangle,$$

then it is not the case that

$$h^\dagger \quad \equiv \quad f^{\dagger\dagger} \quad (\mathrm{mod}\ \Theta),$$

by Theorem 8.1, since $h^\dagger$ has no subtree equal to either $f^{\dagger\dagger}$ or to $g^{\dagger\dagger}$. Thus, the GA-implication fails in $T$.

It remains to prove Lemma 8.4.

## 8.1 Proof of the Lemma

We call a vertex $u$ of $f$ a **zero vertex** if the subtree $f_u$ of $f$ rooted at $u$ is a zero subtree, i.e., $f_u$ is $t \cdot 0_p$, for some $t : 1 \longrightarrow 0$ in $\Sigma\mathbf{tr}$. A tree with no zero vertices is *coaccessible.* A **minimal zero vertex** of $f$ is a zero vertex of $f$ which has no proper prefix which is also a zero vertex. We let $MZ(f)$ denote the set of minimal zero vertices of $f$.

Suppose that $f = F \cdot (\mathbf{1}_p \oplus \alpha) : 1 \longrightarrow p$, where $F : 1 \longrightarrow p + k$ and $\alpha : k \longrightarrow 0$.

1. If $f$ has no zero vertices, then $F$ factors through $p$, i.e., $F = F' \oplus 0_k$, for some $F' : 1 \longrightarrow p$, and $F'$ is coaccessible. If the empty word is a zero vertex, then $F$ factors outside of $p$, i.e., $F = 0_p \oplus F'$, for some $F' : 1 \longrightarrow k$.
2. If $u$ is a zero vertex of $f$ and $uv \in \mathrm{dom}\, f$, then $uv$ is also a zero vertex of $f$.
3. If $u$ is a minimal zero vertex of $f$, then $u \in \mathrm{dom}\, F$. Indeed, otherwise $u = vv'$, where $vF = x_{p+j}$ and $v' \in \mathrm{dom}\, \alpha_j$. But then $v$ is also a zero vertex of $f$, showing $u$ is not minimal.
4. Since $f$ is regular, the collection of trees $\{f_u : 1 \longrightarrow 0 |\ u \in MZ(f)\}$ is finite.

Now assume that $f = F \cdot (\mathbf{1}_p \oplus \beta) = G \cdot (\mathbf{1}_p \oplus \gamma)$. Let $X$ be the (regular) set of all minimal zero vertices $u$ of $f$ such that $F_u$ or $G_u$ contains a leaf labeled by some variable (which is necessarily $x_{p+j}$, for some $j > 0$.) Note that if $u \in X$, then $u \in \mathrm{dom}\, F \cap \mathrm{dom}\, G$, and if $v$ is any leaf of $F_u$, then the label of $v$ is $x_{p+j}$ for some $j > 0$. Indeed, otherwise, $u$ would not be a zero vertex of $f$. Similarly, if $v$ is a leaf of $G_u$ labeled by a variable, its label is $x_{p+j}$ for some $j > 0$.

We let $H$ be $f$ "cut off" at $X$. Assume there are $m$ trees $\delta_1, \ldots, \delta_m$ of the form $f_u : u \in X$.

**Definition 8.6** *The tree $H : 1 \longrightarrow p + m$ is defined as follows. The domain of $H$ is the regular set consisting of $X$ together with the set of all vertices of $f$ having no prefix in $X$. For $u \in \mathrm{dom}\, H$,*

$$uH \quad := \quad \begin{cases} u\, f & \text{if } u \text{ has no prefix in } X \\ x_{p+i} & \text{if } u \in X \text{ and } f_u = \delta_i \\ \text{undefined} & \text{otherwise.} \end{cases}$$

**Remark 8.7** The tree $H$ is regular. If the set $X$ is empty, then $m = 0$.

Now we define the trees $Q : m \longrightarrow k$, $Q' : m \longrightarrow k'$. Suppose that $u \in X$ and $f_u = \delta_i$. Then $u \in \mathrm{dom}\, F \cap \mathrm{dom}\, G$, by item 3. above.

**Definition 8.8**

$$vQ_i \quad := \quad \begin{cases} vF_u & \text{if } vF_u \text{ is not a variable} \\ x_j & \text{if } vF_u = x_{p+j}. \end{cases}$$

$$vQ'_i \quad := \quad \begin{cases} vG_u & \text{if } vG_u \text{ is not a variable} \\ x_j & \text{if } vG_u = x_{p+j}. \end{cases}$$

Then, by construction, for $u \in X$ and $f_u = \delta_i$,

$$f_u \quad = \quad Q_i \cdot \beta \;=\; Q'_i \cdot \gamma.$$

It follows that

$$Q \cdot \beta \quad = \quad Q' \cdot \gamma.$$

Further,

$$\begin{aligned} F &= H \cdot (\mathbf{1}_p \oplus Q) \\ G &= H \cdot (\mathbf{1}_p \oplus Q'). \end{aligned}$$

When $n > 1$ and $f = \langle f_1, \ldots, f_n \rangle : n \longrightarrow p$ can be written in two ways as

$$\langle F_1, \ldots, F_n \rangle \cdot (\mathbf{1}_p \oplus \beta) \quad = \quad \langle G_1, \ldots, G_n \rangle \cdot (\mathbf{1}_p \oplus \gamma),$$

we use the same procedure; we now let $X_i$ be the set of minimal zero vertices $u$ in the tree $f_i$, such that $(F_i)_u$ or $(G_i)_u$ contains a leaf labeled $x_{p+j}$. Let $m$ be the number of all subtrees $(f_i)_u, u \in X_i,\ i \in [n]$. Define the domain of the tree $H_i : 1 \longrightarrow p + m$, $i \in [n]$ as the set of vertices in $X_i$ together with those vertices in the domain of $f_i$ having no prefix in $X_i$; the values of $H_i$ are as above. The trees $Q, Q'$ are defined exactly as before. We omit the remaining details. □

### 8.2 A Generalization

The proof of Theorem 8.2 suggests that the following notion may be of interest. Suppose that $T$ is any theory.

**Definition 8.9** *$T$ has the* **lifting property** *if for any morphisms $f : n \longrightarrow n + p$, $F : n \longrightarrow 1 + p + k$ and $\alpha : k \longrightarrow 0$ in $T$, if*

$$f \cdot (\rho \oplus \mathbf{1}_p) \quad = \quad F \cdot (\mathbf{1}_{1+p} \oplus \alpha),$$

*where $\rho : n \longrightarrow 1$ is the unique base morphism, then there is some $G : n \longrightarrow n + p + k$ such that*

$$\begin{aligned} f &= G \cdot (\mathbf{1}_{n+p} \oplus \alpha) \\ F &= G \cdot (\rho \oplus \mathbf{1}_{p+k}). \end{aligned}$$

It was shown above that $\Sigma\mathbf{tr}$ has the lifting property. We state without proof the following facts.

**Proposition 8.10** *Each matricial theory* [11, 8] *has the lifting property, as does each theory $\mathbf{Pfn}_A$ and $\mathbf{Rel}_A$.* □

**Proposition 8.11** *Suppose that $T$ is an iteration theory which has the lifting property. Then if $T$ satisfies the functorial dagger implication, so does $T/\theta$, where $\theta$ is any zero congruence on $T$.* □

## 9 Acknowledgment

The authors wish to thank L. Bernátsky for his very careful reading of a draft of this paper.

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