Part I

a) (10 points) Fill in the rest of the following table, so that each entry is $O$, $\Omega$, or $\Theta$, according to whether the row function is $O$, $\Omega$, or $\Theta$ of the column function. If more than one is true, you should put the strongest result possible. All logarithms are base 2.

<table>
<thead>
<tr>
<th></th>
<th>$2^n$</th>
<th>$(2n)!$</th>
<th>$n^{n/\log n}$</th>
<th>$2^{n/\log n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^n$</td>
<td>$\Omega$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n^{\log n}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$4^n$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) (15 points) Characterize the order of growth of the following recurrences:

\[ T(n) = 4T(n/2) + O(n) \]
\[ T(n) = 3T(n/4) + O(n^{3/4}) \]
\[ T(n) = 8T(n/4) + O(n^{3/2}) \]

c) (25 points) An $m \times n$ chessboard is to be cut into its $m \cdot n$ unit squares. At each step, the board can be cut either horizontally or vertically. Suppose that the cost of a cut equals the number of unit squares remaining in the smaller of the two resulting sub-boards. For example, performing a horizontal cut on a $2 \times 3$ board results in two $1 \times 3$ sub-boards and thus costs 3, whereas cutting it vertically yields sub-boards of dimensions $2 \times 1$ and $2 \times 2$ and hence costs 2. (Of course, for larger boards, not all horizontal/vertical cuts would have the same cost.) Describe a dynamic-programming algorithm to compute the sequence in which the cuts should be made to minimize the overall cost.

Part II

Consider the CLUSTERING (Clstr) decision problem:

**Instance:** An $n \times n$ symmetric matrix $D$ with non-negative entries (the “distances”), and two non-negative integers $b$ and $k$.

**Question:** Does $D$ allow a $(b,k)$-clustering, that is, does there exist a partition of $\{1, \ldots, n\}$ into $k$ disjoint subsets (or clusters) $X_1, \ldots, X_k$ such that distances within each cluster are bounded by $d$, i.e., $(\forall h \in \{1, \ldots, k\})(\forall i, j \in X_h)[D[i,j] \leq b]$?

a) (5 points) Show that Clstr is in NP.

b) (25 points) Recall the 3-COLORABILITY (3COL) decision problem:

**Instance:** An undirected graph $G = (V,E)$

**Question:** Does $G$ allow a 3-coloring, that is, does there exist a mapping $\sigma : V \to \{0, 1, 2\}$ such that $(\forall (u, v) \in E)[\sigma(u) \neq \sigma(v)]$?

Using the fact that 3COL is NP-hard (which you do not have to prove), show that Clstr is NP-hard.

c) (5 points) Formulate the search version (3-COLORING) of the 3COL decision problem.

d) (15 points) Show a polynomial-time Turing-reduction from 3COL-s to 3COL.