Shape from Shading as a Partially Well-Constrained Problem

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For general objects, and for illumination from a general direction, we study the constraints on shape imposed by shading. Assuming generalized Lambertian reflectance, we argue that, for a typical image, shading determines shape essentially up to a finite ambiguity. Thus regularization is often unnecessary, and should be avoided. More conjectural arguments imply that shape is typically determined with little ambiguity. However, it is pointed out that the degree to which shape is constrained depends on the image. Some images uniquely determine the imaged surface, while, for others, shape can be uniquely determined over most of the image, but infinitely ambiguous in small regions bordering the image boundary, even though the image contains singular points. For these images, shape from shading is a partially well-constrained problem. The ambiguous regions may cause shape reconstruction to be unstable at the image boundary. Our main result is that, contrary to previous belief, the image of the occluding boundary does not strongly constrain the surface solution. Also, it is shown that characteristic strips are curves of steepest ascent on the imaged surface. Finally, a theorem characterizing the properties of generic images is presented. © 1991 Academic Press, Inc.

1. INTRODUCTION

Shape from shading has traditionally been considered an underconstrained problem, with potentially many different surfaces corresponding to a shaded image. As a result, in order to select a unique, "physically reasonable" surface solution for a given image, most algorithms for reconstructing shape have incorporated regularization techniques. For example, the influential variational approach usually includes terms in the objective function enforcing smoothness in the reconstructed surface.

It has been suggested more recently that shape from shading is not necessarily underconstrained when the image contains singular points, i.e. maximally bright points [1]. If this is so, regularization is unnecessary for these cases, and should be avoided, since otherwise it can distort the surface reconstruction. For instance, including smoothness terms in a variational approach can cause the reconstructed surface to be overly smoothed. For this reason, it is important to understand when the constraints on shape from shading are strong enough to render regularization unnecessary. Also, if these constraints can be understood in detail, and explicitly incorporated in a shape reconstruction algorithm, it may be possible to improve the robustness of shape recovery.

In a recent paper, we analyzed the constraints on shape from shading when the illumination is from—or symmetric around—the camera direction [2]. With this lighting, and for general images (modulo standard assumptions), it was proven that shading determines shape uniquely. Thus, a shaded image in this case contains enough information to completely determine the imaged object.

The case of general illumination direction is analyzed in the present paper. As usual, the imaged object is assumed to have uniform, known reflectance, with no specularity. Specifically, we consider surfaces with generalized Lambertian reflectance, for which the observed brightness is independent of the viewing direction. The reflectance function must depend only on the incident angle between the illumination direction and the surface normal, but is not limited to the specific Lambertian functional form. Several constraints on the potential surface solutions are explicitly derived, and it is argued that for a typical image, shading determines shape essentially up to a finite ambiguity. More conjectural arguments suggest that in fact shape is often determined with little ambiguity. However, although shape from shading is typically well-constrained, this is not always true: the strength of the constraint on the surface solution depends on the image. For some images, the reconstruction is uniquely determined. On the other hand, we discuss an explicit example where the surface reconstruction is uniquely determined over most of the image, but infinitely ambiguous within a small image region, despite the presence of singular points. For this image, shape from shading is a partially well-posed problem. We argue that such ill-posed regions can occur frequently, but typically are small fractions of the image.

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2. PROBLEM FORMULATION

We consider a matte surface illuminated by an infinitely distant point source. The image is assumed to be formed by orthographic projection. If the surface is Lambertian, its reflectance function is \( R(p, q) = -\hat{n} \cdot \hat{L} \), and the image irradiance equation is

\[
I = -\hat{n} \cdot \hat{L}. \tag{1}
\]

Here \( I \) is the image intensity, \( \hat{L} \) is the light ray direction, and \( \hat{n} \) is the surface normal. The \( z \) direction represents as usual the depth of 3D points with respect to the image plane; \( p \) and \( q \) are the partial derivatives of \( z \). The surface normal \( \hat{n} \) is defined so that it points towards the observer located at negative \( z \). Explicitly,

\[
\hat{n} = \frac{(p, q, -1)}{(1 + p^2 + q^2)^{1/2}}.
\]

Our results are also valid for more general reflectance functions of the form \( R(p, q) = f(-\hat{n} \cdot \hat{L}) \), where \( f \) is a suitable function, e.g. \( f \) must be strictly monotonic, and \( f(0) = 0 \). For simplicity, however, it will be assumed that the surface is Lambertian in the remainder of the paper.

3. CHARACTERISTIC STRIPS AS CURVES OF STEEPEST ASCENT

The problem of shape from shading is to solve the image irradiance equation, Eq. (1), for the surface \( z(x, y) \) given the intensity \( I \). This is a nonlinear, first order, partial differential equation. The method of characteristic strips is a classical method for the solution of such equations, first applied to the shape from shading problem by Horn [4]. A characteristic strip is, roughly, a line in the image along which the surface depth and orientation can be computed, assuming that these quantities are known at the starting point of the line. A consistent solution to shape from shading determines a flow of characteristic strips in the image, with every image point lying on exactly one characteristic strip line. Conversely, such a flow of characteristic strips uniquely determines a shape solution. The basic approach of our paper is to specify the properties of this flow of characteristic strips, and thus, from the above, the shape solution.

The characteristic strip curves are computed by solving a system of differential equations. In terms of a Hamiltonian function \( H \) [2], where

\[
H(x, y, p, q) = I(x, y) + \hat{n} \cdot \hat{L} = 0,
\]

the characteristic strip equations can be written as

\[
\dot{x} = H_p \left( \frac{\partial H}{\partial p} \right), \quad \dot{y} = H_q, \quad \dot{p} = -H_x, \quad \dot{q} = -H_y. \tag{2}
\]
The dot denotes a derivative with respect to “time,” an arbitrarily chosen variable that parameterizes the position along the characteristic strip. The subscripts denote partial differentiation. Explicitly,

\[ \dot{x} = H_p = \frac{L_x}{(1 + p^2 + q^2)^{1/2}} - p \left( \frac{\hat{n} \cdot \hat{L}}{1 + p^2 + q^2} \right) \]

\[ \dot{y} = H_q = \frac{L_y}{(1 + p^2 + q^2)^{1/2}} - q \left( \frac{\hat{n} \cdot \hat{L}}{1 + p^2 + q^2} \right) \]

Also,

\[ \dot{z} = p \dot{x} + q \dot{y} \]

by the definition of \( p, q \). After some algebra, and a redefinition of the arbitrary time parameter, the above equations may be rewritten as

\[ \frac{dr}{dt} = \hat{L} - (\hat{n} \cdot \hat{L})\hat{n}. \quad (3) \]

A curve generated by this equation is on the surface of the illuminated object, and will be referred to as a surface strip. Note our convention for the time direction: as time increases, the trajectory heads away from the light source.

The form of this equation makes it clear that the surface curve corresponding to a characteristic strip depends only on the 3D object and the illumination direction, and not on the viewing direction. In other words, if the lighting is kept constant, characteristic strips in different images of an object represent the identical surface strips. Also, the right-hand side of Eq. (3) is just \( \hat{L} \) projected into the surface tangent plane. Thus surface strips are curves of steepest ascent in the \( \hat{L} \) direction. Finally, it is clear that the surface strips can be extended smoothly beyond the illuminated and visible regions in accordance with Eq. (3), so as to cover the whole of the object.

4. SINGULAR POINTS

The importance of singular points in determining and fixing a solution to the shape from shading problem has been stressed by many people [1, 3, 5], and they are crucial in this work as well. The singular points of an image are conventionally defined as those at which the value of the intensity \( I \) is maximal for the given reflectance function. Their importance stems from the fact that the surface orientation is uniquely determined at these points. Also, suppose we have a cartesian coordinate system \( (\xi_1, \xi_2, \xi_3) \), where \( \xi_3 = \hat{L} \), i.e., the third coordinate measures distance along the light ray direction. Then the singular points in an image of a surface \( S \) just correspond to the critical points of \( \xi_3 \) on \( S \) (e.g., the maxima or minima). The object points corresponding to singular points are independent of the view, as with characteristic strips, and depend only on the illumination direction.

We will be mostly concerned with nondegenerate singular points. A singular point is defined to be nondegenerate if the matrix of second derivatives of the intensity \( I \) is nonsingular at the point. At a nondegenerate singular point, the principal curvature values of the surface are essentially determined, and are nonzero (assuming the second derivatives of \( I \) satisfy a certain additional technical condition, as is generically the case). Thus, the surface at such a point must be either convex, concave, or saddle-shaped.

Corresponding to the three possibilities for the surface, the characteristic strip flow in the neighborhood of a nondegenerate point can be classified into just three possibilities. This follows from the Grobman–Hartman theorem (see, e.g., [6]), and was first noted by Saxberg [3]; it also clearly follows from the fact that surface strips are curves of steepest ascent. If the surface at the singular point is convex, then the direction of all characteristic strips near the point is outward, and the point is referred to as a source. Similarly, a concave surface corresponds to a sink, with all nearby strips converging to the sink. These singular points will be referred to collectively as elliptical. For a saddle singular point, the flow is more complicated. Each saddle point is the originating point for two characteristic strips, and similarly the terminating point for two. Other characteristic strips approach, and then recede from the saddle point, without actually connecting to it [2, 7].

The importance of nondegenerate singular points for constraining the surface solutions follows from the local uniqueness theorem of Bruss [5]. The theorem states that, near a nondegenerate singular point, there exist a unique surface solution that is convex at the point, and a unique surface solution that is concave there. Although Bruss proved this theorem assuming illumination from the camera direction, it was later realized by Saxberg [7] that the result was valid for illumination from a general direction. It is argued below that the convex or concave solution regions associated with these singular points typically cover most or all of the image, thus constraining the surface solution. In [2], for illumination from the camera direction, it was proven that these regions always cover the image, and determine the solution uniquely.

5. THE FLOW OF CHARACTERISTIC STRIPS

In this section we briefly derive some basic properties of the image flow of characteristic strips using the fact that they correspond to steepest ascent curves on the imaged object. In the next section, the constraints on the
object solutions due to these results are discussed. Many of these results are direct generalizations of ones derived in [2], where they are treated in more detail.

Only images with a finite number of nondegenerate singular points are considered. This is not a real restriction; it will be shown later that such an image can always be obtained by an infinitesimal perturbation of the imaged surface, and that essentially all images have this property. In other words, these images constitute a generic class.

Characteristics Strips Terminate at Singular Points. Consider a surface corresponding to an image as above. Every point on the surface clearly lies on a unique curve of steepest ascent. Correspondingly, every point in the image lies on a unique characteristic strip line. The projection of a characteristic strip in the image will be referred to as a base characteristic. We can assume that the surface is bounded, and compact. Then every steepest ascent curve must terminate at both ends. Since these curves ascend in the direction of the illumination \( L \), it is clear that they must terminate at critical points of the distance function \( \xi_3 \)—for instance, at a local maximum—or else on the object boundary. (As before, the function \( \xi_3 \) measures distance along the \( L \) direction.) Since singular points correspond to critical points of \( \xi_3 \), the surface has a finite number of these. Therefore every steepest ascent curve terminates at both ends either at a unique critical point, or else on the object boundary. Correspondingly, every characteristic strip terminates at each end either at a singular point, or at the image boundary [2]. If an image point \( p \) lies on a strip connecting to a singular point \( s \), then \( p \) and \( s \) will be said to be connected.

Also, by the Groban–Hartman theorem, every saddle point connects to just four characteristic strips. Thus, only a finite number of characteristic strips in the image connect to saddle points. Essentially all strips connect to either a source or sink, or else the image boundary, at both ends. It is suggested in the next section that the region of strips connecting just to the boundary is likely to be a small fraction of the image. Then, since the surface is essentially determined at every image point connected to an elliptical point (see below), this implies that the surface may be determined over a large fraction of the image.

Connectivity of Critical Points. To derive additional results, we now extend the imaged object to a closed surface \( S \). Clearly, this can always be done if the extension is transparent.

Lemma. For a closed surface with a finite number of critical points, every critical point is connected by a sequence of steepest ascent curves to every other critical point. This result is useful since a characteristic strip connecting two singular points determines their relative depth [2]. Also, the nature of the solution at a singular point, whether convex, concave, or saddle, can be constrained if a strip from a singular point of known type connects to it [2].

Proof Sketch. This result can be proven completely generally along the lines of the arguments in [2]. However, it is easier to prove assuming structural stability of the surface. Structural stability, defined below, implies that there exists no steepest ascent curve (or characteristic strip in the image) connecting two saddle points. It is proven in the appendix that essentially all surfaces are structurally stable [2].

Suppose that the critical points divide into two sets \( C \) and \( N \), which do not connect to each other. Since a surface is composed of an infinite number of steepest ascent curves, it must contain at least two elliptical critical points which act as originating and terminal points for these curves. These are the surface points respectively closest to and furthest from the light source. Therefore, we may assume that \( C \) contains a set of elliptical critical points, denoted by \( C_e \). Let \( U_e \) be the set of all points connecting to the points of \( C_e \). \( U_e \) is the union of (un)stable manifolds, and therefore an open set [2, 6]. Steepest ascent curves passing through points on the boundary of \( U_e \) must be contained in the boundary, by continuity of the flow. Every such curve terminates at two critical points, which must also be on the boundary, again by continuity of the flow. Thus, the boundary consists of critical points, and possibly steepest ascent curves joining critical points. If the boundary only contained isolated critical points, then the closure of \( U_e \) would be the whole surface, implying \( N \) is empty contrary to assumption. Thus we can assume that the boundary of \( U_e \) contains steepest ascent curves.

Consider the terminating points of one of these curves. Since the surface is structurally stable, at least one must be elliptical. However, an elliptical point is characterized by the fact that any steepest ascent curve within some neighborhood terminates at the point. Thus, an elliptical point on the boundary of \( U \) must actually be connected to some critical point in \( C_e \), and therefore contained in \( C_e \) by the definition of this set. But this is impossible, since it is on the boundary of \( U_e \).

Q.E.D.

When the illumination is from the camera direction, this result implies that every singular point in the image is connected by a sequence of characteristic strips to every other singular point [2]. For general illuminant direction, this need not be the case, as the steepest ascent curves connecting singular points in the image may be partly invisible.

Structural Stability and Generic Surfaces. A closed surface \( S \) is defined to be structurally stable with respect to the height function \( \xi_3 \) if (1) the surface has a finite
number of critical points, (2) they are all nondegenerate, (3) there are no steepest ascent curves connecting saddle critical points. These surfaces can be shown to be stable in the sense that the flow of steepest ascent curves does not change drastically when the surface is perturbed—its topology remains the same. In the Appendix the following theorem is proven, extending a similar result of [2]:

**Theorem.** (1) Every smooth, closed surface can be approximated arbitrarily well by a structurally stable surface. (2) For all sufficiently small perturbations of a structurally stable surface, the perturbed surface is also structurally stable. This theorem is a corollary of the Palis–Smale theorem [8].

A generic class is one containing essentially all instances, apart from a few special cases. If an instance is contained in this class, then all instances within some neighborhood are also. Moreover, for any special case not contained in this class, an infinitesimal perturbation will yield an instance contained in the class. From the theorem above, the structurally stable objects form a generic class of objects.

For an image of a structurally stable object, the above results imply: (1) the number of singular points in the image is finite, (2) every singular point is nondegenerate, and (3) there is no characteristic strip connecting two saddle-type singular points. The last property is valid for the flow of strips corresponding to the structurally stable object. Since essentially all objects are structurally stable, these properties will hold for essentially all images.

**Base Characteristics at the Limb.** The limb is defined to be the image of the occluding boundary. We can assume that it is a smooth curve in the image, since this is generically true for a non-self-occluding surface [9]. It can be distinguished from an edge boundary by the property that the derivative of the intensity becomes singular at the limb [3]. For points on the occluding boundary, the surface tangent plane contains the viewing direction \( \hat{z} \), and also the tangent to the limb in the image plane. A steepest ascent curve passing through the occluding boundary has a well-defined 3D tangent which must lie in this tangent plane. Therefore, unless this tangent has zero projection in the image plane, the corresponding characteristic strip at the limb will be tangent to the limb. In general, therefore, characteristic strips in a consistent solution are tangent to the limb. This is shown directly in Section 7. The direction of a strip at the limb may be either into or out of the image region (see Section 8).

**Existence of a Source in the Image.** Below, we assume that backlighting is excluded. Thus, the light source and camera are in the same hemisphere with respect to the viewed object. Also, we assume as above that the object is a closed surface, contained in the field of view. Then, there is at least one source singular point in the image, corresponding to the object point closest to the light source.

**Characteristic Strips at the Shadow Boundary.** The shadow boundary is the boundary on the object between the illuminated and shadowed regions. Its hallmark is that the intensity falls smoothly to zero at this boundary. Along the boundary \( \hat{n} \cdot \hat{L} = 0 \), and therefore the tangent to any base characteristic at this boundary is just the projection of \( \hat{L} \) into the image plane, from Eq. (3). (Note that even though their tangent is determined at the shadow boundary, this is not sufficient to determine the characteristic strips or surface there.) In [2] it was shown that for a closed surface, the surface is convex at the occluding boundary; that is, it rolls away from the viewer. The same argument shows that the surface is convex at the shadow boundary, since this is nothing more than the occluding boundary from the viewpoint of the light source position. Thus, steepest ascent curves, since they ascend in the direction away from the light source, can only exit the illuminated region towards the shadowed region [2]. This result is useful because as a consequence the flow direction of a base characteristic intersecting the shadow boundary is determined.

**The Poincare–Hopf Index Theorem.** For a closed surface, this theorem states that

\[
N_{el} - N_{sad} = E,
\]

where \( N_{el} \) is the number of elliptical and \( N_{sad} \) the number of saddle critical points, and \( E \) is the Euler number of the surface. For a genus zero surface, \( E = 2 \). If the total number of critical points is known, this relation determines the number of saddles. See [2, 10].

6. **CONSTRAINTS ON THE SHAPE SOLUTION**

In this section the constraints on the shape solution are discussed. First, a preliminary result is needed. Recall from Section 4 that the surface is uniquely determined near a concave or convex singular point (up to an overall translation if the depth of the singular point is not known.) Moreover, the surface is determined everywhere along a characteristic strip assuming it is known at the initial point of the strip. Thus, the surface is determined along every strip connected to a known elliptical singular point [2].

We now consider the simplest case of an image containing exactly one singular point \( s \). It is assumed as before that the imaged surface is closed, and completely contained within the field of view. Note that this assumption is in fact a restriction on the image. It requires that the image boundary be composed of limb segments and/or shadow boundaries—either the intensity falls
smoothly to zero at a point on the image boundary, or else the derivative of the intensity is singular there. Conversely, if the intensity always satisfies one of these two properties at the image boundary, then the image boundary must be composed of limb and/or shadow boundary segments. Below, the pre-image of the image boundary—i.e., the curve on the object bounding the imaged region—is referred to as the visibility boundary.

Assuming also that backlighting is excluded, $s$ must be a source. The base characteristics originating at $s$ terminate at the image boundary, and must cover at least some fraction of the image; the unique convex surface solution at $s$ can be extended uniquely over this region. Thus, for such images, the surface solution is determined uniquely over some fraction of the image.

The solution is uniquely determined over the whole image if the base characteristics originating at $s$ cover the whole of the image boundary. Proof: A base characteristic passing through an arbitrary point in the image originates at a singular point or on the boundary. It cannot originate at the boundary, since all strips at the boundary originate at $s$. Thus, it must originate at $s$, and the solution is therefore uniquely determined along this strip.

Conversely, if segments of the image boundary are not covered by strips originating at $s$, then the solution may be uniquely determined only over part of the image, as is demonstrated explicitly in Section 8. In this case, for any consistent surface solution, some of the base characteristics (and corresponding surface strips) must be entering the visible region at the image boundary. Since, by the reasoning of Section 5, this does not happen at the shadow boundary, it must occur at the occluding boundary. Let $P$ be a point on the occluding boundary such that on one side the surface strips are exiting, while on the other they are entering the visible region. The surface strip $C_P$ passing through $P$ must be tangent to the occluding boundary in three dimensional space. By continuity of the flow, it originates at the source $s$, and reenters the visible region following $P$, finally exiting this region at another point. The conclusion is that the solution may be nonunique only when a base characteristic from $s$ intersects the image boundary twice. The region of the image where the solution is potentially ambiguous is bounded by the image boundary on one side, and a base characteristic with this property on the other. This is shown in Fig. 1. Moreover, again by continuity, there is a surface strip $C_3$ close to the one just described, which originates at $s$, exits the visible region at the occluding boundary near $P$, loops around $P$, reenters the visible region at a point on the occluding boundary on the other side of $P$, and finally exits elsewhere. For nonuniqueness, there must therefore exist a surface strip that intersects the occluding boundary twice, and the visibility boundary at least three times.

Recall, however, that surface strips are curves of steepest ascent in the light ray direction $\hat{L}$. In the case above, the occluding boundary intersects one of these surface strips twice. The complete visibility boundary intersects this strip a third time, possibly at the shadow boundary. Thus, the occluding boundary and the visibility boundary must also be ascending rapidly in the light ray direction—in fact, more quickly than the surface strip over some interval. For many objects which are not too extended in depth, the occluding and shadow boundaries probably do not ascend sharply in the $\hat{L}$ direction for very long. If the viewing and illumination directions are not too dissimilar, this conclusion increases in likelihood. Thus, one might expect that the potentially ambiguous image regions are typically small fractions of the image. This is the case in the examples considered so far, including the surface of Fig. 19, Section 8, which was contrived to have a (relatively) large ambiguous region. This suggestion for the typical case should be confirmed by the study of a large number of images of a variety of objects.

If our conjecture is valid, then typical shaded images containing one singular point uniquely determine shape over most of the image. Next, we apply this conjecture to more general images. If it remains valid, then the regions connecting just to the image boundary will tend to be small. From the previous section, this implies that most of the image connects to elliptical singular points. Therefore, the shape solution is determined over most of the image up to (1) the finite ambiguity of determining which singular points are sources and sinks, and (2) possible ambiguities in patching together the solutions around different elliptical points.

Let us assume first that the nature of the solution at every singular point is known, and analyze the remaining ambiguity. To begin, we characterize the potentially ambiguous regions that do not connect to elliptical singular points in the image. Let $B$ denote the region of all image points connecting at both ends to the image boundary, let $B$ be the closure of $B$, and let $B_1$ denote the interior of $B$. $B_1$ comprises the potentially ambiguous regions of the image, as indicated in Figs. 1 and 2, for example. A point on the boundary of $B$ which is not on the image boundary must either be a singular point, or else lie on a base characteristic. If it is a singular point, it must be a saddle, since in some neighborhood of an elliptical point all characteristic strips connect to it. If it lies on a base characteristic, then (assuming the surface is structurally stable as is generically the case) this characteristic must connect to the image boundary at least at one end, since it cannot connect to saddles at both ends. At the other end, it may either connect again to the image boundary, or else to a saddle. In the first case, the base characteristic is of the type described for a one singular point image—it corresponds to a surface strip that at one point is tangent
to the occluding boundary, and subsequently reenters the visible region. In the second case, then there is also a second strip connecting to the saddle which is on the boundary of $B$ [2]. Again by structural stability, this second strip connects to the image boundary. The two strips represent adjacent "arms" of the saddle, embracing between them a quadrant containing strips from $B$. An explicit example of such a region of $B$ is presented later in Fig. 20, Section 9. Also, on the line obtained by concatenating the two strips, the distance from the light source increases monotonically, just as it does along any one characteristic strip [2].

The conclusion is that the potentially ambiguous regions are bounded by (1) image boundary segments, (2) base characteristics connecting at both ends to the image boundary of the type described above, and/or (3) a line consisting of a saddle point, and two base characteristics connecting this saddle to the image boundary. The interior bounding lines of types 2 or 3 always terminate at both ends on the image boundary. Note that our conjecture applies to a bounding line of type 3 as well as to one of type 2, because of the monotonicity property above. Therefore, these bounding lines should typically be short. Examples of the different bounding lines are shown in Fig. 2.

For simplicity, we now assume that the image has no holes, i.e., that the region where $I > 0$ is simply connected. The characteristic strips connecting to the sources and sinks are uniquely determined, as proven above. Consider the complement $C_I$ of $B_I$. It contains the elliptical points of the image, and all points lying on strips connecting to them. Consider a connected region $R$ of $C_I$ (Fig. 2). It must also be simply connected; that is, it can contain no holes or islands of $B_I$, since every point in $B_I$ connects to the image boundary.

We claim that the relative depths of all elliptical points in $R$ are determined; this implies by continuity that the surface solution is determined over the whole region $R$.

**Proof Sketch.** Consider some elliptical point $e$, and the region of points connected to $e$. The depths of points on the boundary of this region are determined relative to the depth of $e$ by continuity. Thus the relative depth of two elliptical points is determined if the regions connected to them share a boundary point. Consider the set of all elliptical points in $R$ whose relative depths are determined in this way. Let $U$ consist of all points connected to these elliptical points, and let $\overline{U}$ denote its closure. If there are additional elliptical points in $R$, then $\overline{U}$ has a boundary in $R$. But this boundary must be shared with some of the additional elliptical points, whose depths are therefore determined relative to the points of $\overline{U}$. This contradicts the assumption that the elliptical points of $U$ form a maximal set in the sense above. Thus, the relative depths of all elliptical points in $R$ are determined.

Q.E.D.

Finally, if our conjecture above is correct, then for a typical image the potentially ambiguous region $B_I$ will be small, consisting of slivers of the image bordering the image boundary. The complement $R$ will typically be a single connected region covering most of the central image. Thus, assuming that the nature of the singular points is known, the result above shows that the surface solution is uniquely determined over most of the image in the typical case.

To what extent is the nature of the solution at each singular point determined? Our results on this question are preliminary, but several constraints can be adduced. Suppose $e$ is known to be a source or sink, and let $U$ be the region connected to $e$. Then the nature of the solution at all singular points on the boundary of $U$ is determined [2]. Note that if this boundary does not contain singular points, or contains only saddle singular points, then (for a structurally stable surface) it consists of bounding lines of type 2 or 3 above connecting at both ends to the image boundary (plus image boundary segments). But by our conjecture such lines should typically be short, and the ambiguous regions bounded by them and the image boundary small. Therefore, in this case $U$ would cover most of the image, and the solution would anyway be
determined over most of the image. Typically, therefore, there should be at least one elliptical point on the boundary of \( U \). From the result just quoted, this point is necessarily elliptical: moreover, its solution type (convex vs. concave) is uniquely determined [2].

Applying the result again, the elliptical singular points on the boundary of \( U \) may in their turn determine the nature of the solution at yet other singular points. By a chain reaction of this reasoning, it is possible that the type of many singular points may be determined. Thus, typically, there may be little ambiguity in assigning the solution type at each singular point. In combination with the previous arguments, this suggests that the surface solution may be almost uniquely determined over most of the image in the typical case.

Another constraint has already been mentioned, namely the shadow boundary. At this boundary the direction of a characteristic strip is outwards, so that an elliptical point connecting to it must be a source.

However, we conjecture that the most important constraint follows from the potential “impossibility” of images. It was argued in [2] that most images (considered simply as intensity functions \( I(x, y) \)) are impossible, in the sense that there is no consistent corresponding surface of which they could be the image (see also [11, 12]). Moreover, any true image of a surface can be modified by a small perturbation of its intensities so that it becomes impossible. This was shown for illuminant direction close to the viewing direction, but is probably true in general. The basis of this impossibility is that the intensity perturbation can introduce crossing of base characteristics in the image plane; this is impossible since it would imply that there are two steepest ascent curves passing through a single object point. We conjecture that most choices for the singular point types simply do not lead to a consistent surface solution. Recall that if a singular point is taken to be a source or a sink, then the characteristic strips connecting to it are uniquely determined. If a singular point is wrongly identified as a source or sink, then it appears very likely that the strips connecting to it will intersect strips from other elliptical points, implying that the choice of solution type for this point was incorrect. This conjecture will be explored in more depth in future work.

One caveat is that in practice the above constraint may be rather subtle. In the presence of noise and discretization error, which are also potential sources of impossibility, it is not clear how useful the impossibility constraint will remain. In a variational approach, perhaps this constraint would imply that for the wrong choice of singular point type, the minimized objective function corresponding to this choice would be higher than for the correct choice—that is, this choice would yield a local but not an absolute minimum for the objective function.

7. SHAPE RECONSTRUCTION NEAR THE LIMB

In this section, the characteristic strip equations near the limb are examined, and it is shown that the limb does not constrain the shape solution. This is the main theoretical result of the paper. The difficulty in doing this stems from the singularity of the variables \( p \) and/or \( q \) at the limb. Thus we begin by rewriting the equations in a rotated coordinate system, in terms of variables that remain finite on the limb. This strategy of using a rotated coordinate system has also been explored by Saxberg [3].

The characteristic strip equations are Eq. (3) above and

\[
\dot{p} = - \frac{\partial I}{\partial x} (1 + p^2 + q^2)^{1/2}, \quad \dot{q} = - \frac{\partial I}{\partial y} (1 + p^2 + q^2)^{1/2},
\]

(4)

with the surface represented by \( z(x, y) \), \( p \), \( q \). Instead, we switch to representing the surface by a function \( y(x, z) \), and \( y_x, y_z \), which should be possible at least locally. These quantities indeed remain finite on the limb.

Define the variables

\[
w = - \frac{p}{q}, \quad v = \frac{1}{q}.
\]

(5)

If a surface solution \( y(x, z) \) exists, then \( w = y_x \) and \( v = y_z \). These variables evolve via

\[
\dot{w} = \frac{-\dot{p}}{q} + \frac{p \dot{q}}{q^2} = \frac{v}{|v|} (I_x + wI_y)(1 + w^2 + v^2)^{1/2},
\]

\[
\dot{v} = - \frac{\dot{q}}{q^2} = \frac{v}{|v|} (vI_x)(1 + w^2 + v^2)^{1/2},
\]

(6)

The existence of a surface solution \( y(x, z) \) also implies that

\[
I_x = \frac{\partial I}{\partial x}, \quad I_y = \frac{\partial I}{\partial y}, \quad I_z = \frac{\partial I}{\partial z} |_{x}, \quad y_x I_y, \quad y_z I_y,
\]

(7)

where on the left-hand side \( I(x, y = y(x, z)) \) is thought of as a function of \( x, z \). Although \( I_x \) or \( I_y \) must be singular on the limb, \( I_x \) and \( I_z \) are both finite. Equations (6) appear to have almost the same form in the rotated coordinate system as they did in the original one, with \( y \) playing the role of \( z \).

At least one surface solution of the characteristic strip equations can be assumed to exist—namely the original object from which the intensity image was derived. The basic strategy of our proof is to write the equations with reference to this “pre-existing” solution. Below, therefore, \( y_x \) and \( y_z \) denote the partial derivatives correspond-
ing to this reference solution. Similarly, $I'_x$ and $I'_z$ denote the partial derivatives of the intensity assuming the reference solution. These quantities should be thought of as fixed functions of $x$ and $y$, like the intensity $I$ itself. They have the nice property that they remain finite on the limb.

One has

$$I_y = \frac{I'_x}{y_z}, \quad I_z = I'_x - I'_z \frac{y_x}{y_z}.$$  \hspace{1cm} (8)

Substituting into Eqs. (6) yields

$$\dot{w} = \text{sign}(v) \left( I'_z + I'_x \frac{w - y_x}{y_z} \right) (1 + w^2 + v^2)^{1/2},$$

$$\dot{v} = \text{sign}(v) v \frac{I'_z}{y_z} (1 + w^2 + v^2)^{1/2}. \hspace{1cm} (9)$$

The surface normal can be written as

$$\hat{n} = \text{sign}(v) \frac{(-w, 1, -v)}{(1 + w^2 + v^2)^{1/2}}. \hspace{1cm} (10)$$

Since $\hat{n}$ is expressed as a function of $w$ and $v$, the system of Eqs. (3) and (9) form a complete set that can be solved for $x$, $y$, $z$, $w$, $v$ as functions of time. For the reference solution, the surface normal $n'_x$ has the same form as Eq. (9) with $(v, w)$ replaced by $(y_z, y_x)$. Since the limb is defined by $n'_x = 0$, it follows that $y_z = 0$ on the limb, and only on the limb. The singularity of Eqs. (9) is due solely to this vanishing of $y_z$.

What is the significance of this singularity? To answer this question, we consider the signs of the various quantities in the above equations. Let us restrict our considerations to a small portion of the image bordering the limb. Image plane coordinates are chosen so that the limb is tangent to the $x$ direction at some point; this point can be taken to be the origin. For convenience, we assume that $y$ attains a local maximum at this point, so that the limb is convex there. Also, it is assumed that the reference surface is “rolling away” at the limb, as discussed in Section (5).

In the evolution equation for $v$, the variation of $v$ is proportional to $v$ itself. Thus $v$ never changes sign except at singularity points of the right-hand side, namely on the limb. Therefore, $v$ and $y_z$ can be chosen positive over the given region, and the sign factors in Eqs. (9) and in $\hat{n}$ can be neglected. Henceforth, we will drop the sign factors from these equations.

The sign of $I'_z$ is assumed negative over the given region. From the equation for $v$ in Eqs. (9), this implies that $v$ decreases with increasing time, and similarly for $y_z$ in the reference solution. Thus, this choice of sign implies that the direction of the base characteristics is outward at the limb, toward the invisible region.

An explicit solution for $v(t)$ is

$$v(t) = v_0 \exp \left( -\int_{t_0}^t dt \frac{I'_z}{y_z} (1 + w^2 + v^2)^{1/2} \right). \hspace{1cm} (11)$$

Clearly, because $y_z$ goes to zero at the limb, $v$ is forced toward 0 as a base characteristic approaches the limb. Similarly, $w$ is forced toward $y_x$.

The possibility of multiple solutions to shape from shading arises from the following fact: although $v$ and $w - y_x$ must scale to zero with $y_z$ near the limb, their absolute scale is not determined. Thus, we again define new variables

$$s = \frac{v}{y_z}, \quad r = \frac{w - y_x}{y_z}. \hspace{1cm} (12)$$

The time evolution equation for $s$ is

$$\dot{s} = \frac{s}{y_z} (I'_z D^{1/2} - \dot{y}_z), \hspace{1cm} (13)$$

where

$$D = 1 + w^2 + v^2 = 1 + y_z^2 s^2 + (y_z r + y_x)^2. \hspace{1cm} (14)$$

Since

$$\frac{\partial}{\partial x} \bigg|_y = \frac{\partial}{\partial x} - \frac{y_x}{y_z} \frac{\partial}{\partial y} - \frac{1}{y_z} \frac{\partial}{\partial s}, \hspace{1cm} (15)$$

the time evolution of $y_z$ is

$$\dot{y}_z = \dot{x} \left( y_{zx} - \frac{y_x}{y_z} y_{zz} \right) + \dot{y} \frac{y_{zz}}{y_z}. \hspace{1cm} \text{Substituting Eq. (3) into this equation yields}$$

$$\dot{y}_z = \frac{1}{y_z} \left( (y_{zx} y_z - y_x y_{zz})(L_x - I \frac{y_z r + y_x}{D^{1/2}}) \right. \hspace{1cm} \text{Substituting Eq. (3) into this equation yields}$$

$$+ \left. y_{zz} \left( L_y + \frac{I^{1/2}}{D} \right) \right). \hspace{1cm} (16)$$

From the equation for $I'_z$, Eq. (1),

$$I'_z = \frac{(y_{zx}, 0, y_{z2})}{D^{1/2}} \frac{\dot{L}}{D} (y_x y_{z2} + y_z y_{zz}), \hspace{1cm} (17)$$

where

$$D_r = 1 + y_z^2 + y_{z2}^2. \hspace{1cm} (18)$$
Combining the above equations,

\[
\dot{s} = \frac{S}{\sqrt{y_z}} \left[ y_z \left( \frac{D}{D_r^{1/2}} \left( y_z L_x + y_z L_z \right) - \frac{y_z D_{1/2}}{D_r} I(x_y y_x + y_z y_z) \right) - (y_{zz} y_z - y_x y_{zz}) \left( L_x - \frac{y_{zz} y_z + y_z}{D^{1/2}} \right) - y_z \left( L_y + \frac{I y_z y_z}{D^{1/2}} \right) \right].
\]  

(19)

The \( y_{zz} \) terms within the square brackets combine to give

\[
y_z \left( L_z y_z \left( \frac{D^{1/2}}{D_r^{1/2}} + y_x L_x - y_x (y_z r + y_z) \frac{I}{D^{1/2}} \right) - \left( L_y + \frac{I}{D^{1/2}} \right) - I y_z \frac{D_{1/2}}{D_r} \right) = y_{zz} A.
\]  

(20)

Since

\[
y_z L_x + y_z L_z = L_y + \frac{I D_{1/2}}{2},
\]  

(21)

\( A \) can be rewritten

\[
L_z y_z \left( \frac{D^{1/2}}{D_r^{1/2}} - 1 \right) + I \left( D_r^{1/2} - \frac{1 + y_z^2 + y_x y_x r}{D^{1/2}} - y_z^2 \frac{D^{1/2}}{D_r} \right).
\]  

(22)

Similarly, the \( y_{xz} \) terms in the square bracket combine to

\[
y_{xz} y_z \left( L_x \left( \frac{D^{1/2}}{D_r^{1/2}} - 1 \right) - y_z \frac{D_{1/2}}{D_r} - \frac{1}{D^{1/2}} \right) + I y_x r \frac{D_{1/2}}{D_r} = y_{xz} B.
\]  

(23)

It is easy to check from the expressions above that the terms in square brackets are of order \( y_z^2 \) as \( y_z \to 0 \) on the limb. Thus the evolution equation for \( s \) is well defined at the image boundary.

In fact, on the limb,

\[
A = \frac{L_z y_z r}{1 + y_z^2} + \frac{I}{2(1 + y_z^2)^{1/2}} \left( r^2 + \frac{1 + y_z^2}{y_z} - 2 \right),
\]  

(24)

and

\[
B = \frac{L_z y_z r}{1 + y_z^2} + \frac{I r}{(1 + y_z^2)^{1/2}} \frac{1 - y_z^2}{1 + y_z^2}.
\]  

(25)

The equation for \( r \) can be derived similarly:

\[
\dot{r} = \frac{I r D_{1/2}}{y_z} - \frac{\hat{y}_z}{y_z} \left( I_r D_{1/2} - \hat{y}_z \right).
\]  

(26)

The second term has already been examined above, and shown to have a well defined limit on the limb. In the first term, \( I_r \) is given by the expression for \( I_r \) above, Eq. (17), with \( y_{x}, y_{xx} \) replaced by \( y_{x}, y_{xx} \), respectively. Making the same replacements in Eq. (16) for \( \hat{y}_z \) yields an expression for \( \hat{y}_x \). Thus

\[
\dot{r} = \frac{1}{y_z^2} \left( A(y_z r + y_{zz}) + B(y_{zz} r + y_{xx}) \right).
\]  

(27)

As before, \( A/y_z^2 \) and \( B/y_z^2 \) are well defined on the occluding boundary as functions of \( y_z, y_x, r, \) and \( s \).

\( y_z \) is not a continuously differentiable function of the image plane coordinates \( x \) and \( y \). Thus we replace the variable \( y \) by \( z \), \( x, y \) the value of \( z \) for the reference solution given \( x, y \). The differential equation for the time evolution of \( z \) is

\[
\dot{z} = -\left( \frac{y_x}{y_z} \right) \left( L_x - \frac{I}{D^{1/2}} \left( y_z + y_x \right) \right) + \frac{1}{y_z} \left( L_y + \frac{I}{D^{1/2}} \right)
= L_z + \frac{y_x r}{D^{1/2}} + \frac{I}{y_z D^{1/2}} \left( -D^{1/2} D_{1/2} + 1 + y_z^2 \right).
\]  

(28)

Again, it is clear that this is well defined on the limb when \( y_z = 0 \). The system of equations for \( P = (x, z, s, r) \) is consistent and differentiable everywhere including on the limb. Let the limb considered as a curve in the \( x-y \) plane be parameterized by \( \sigma \). For arbitrary choice of initial conditions for \( s, r \) on the limb, the usual theorems of differential equations state that there exists a unique solution of the system of differential equations, with \( P \) a differentiable function of \( \sigma \) and \( t \). Given this solution, \( y \) and \( z \) can also be computed by simple integration of their equations of motion.

It now remains to show that \( w = y_x, v = y_z \), and that the parametric form expressing \( x, y, z \) in terms of \( t \) and \( \sigma \) can be converted into a surface function \( y(x, z) \) or \( z(x, y) \). Consider \( U = \dot{y} - w \dot{x} - v \dot{z} \). By direct computation, using the expressions above for the time derivatives, one can show that \( U \) vanishes everywhere. Next consider \( V = y_x - w x - u z \). We want to demonstrate that this also vanishes. Define

\[
V_o = \frac{V}{u} = q y_x + p x - z, \quad U_o = \frac{U}{u} = q \dot{y} + p \dot{x} - \dot{z} = 0.
\]  

(29)

(30)

Away from the limb,

\[
\frac{\partial V_o}{\partial t} - \frac{\partial U_o}{\partial \sigma} = \frac{\partial V_o}{\partial t} = p \dot{x} - \dot{p} x - q \dot{y} - q y_x = \frac{\partial H}{\partial \sigma} = 0.
\]  

(31)
(Since \(0 = H = I + \dot{n} \cdot \dot{L}\) is true on the limb, and preserved by the differential equation, it is true for all \(t, \sigma\).) Thus

\[ V = uf(\sigma), \]

for some function \(f(\sigma)\) which is independent of time. This is true on the limb as well by the smoothness of the solutions to the differential equations. We require \(V = 0 = f(\sigma)\), which is equivalent to requiring \(\dot{V} = 0\) everywhere on the limb. Explicitly, on the limb,

\[
\frac{\partial V}{\partial t} = (\frac{\partial y_x}{\partial \sigma}, 0, \frac{\partial y_z}{\partial \sigma} (= 0)) \cdot \dot{L} - \frac{I}{(1 + y_x^2)^{1/2}} y_x \frac{\partial y_x}{\partial \sigma}
\]

\[ - (I' + I''\sigma)(1 + y_x^2)^{1/2} \frac{\partial x}{\partial \sigma} - sI'(1 + y_x^2)^{1/2} \frac{\partial z}{\partial \sigma}. \]

Requiring this to vanish places one constraint on the choice of boundary conditions for the triple \((r(\sigma), s(\sigma), R)\), with

\[ R = \frac{\partial z}{\partial \sigma} / \frac{\partial x}{\partial \sigma}. \]

If the boundary conditions obey this constraint, \(U = V = 0\) everywhere as desired.

As long as the Jacobian

\[
\begin{vmatrix}
  x_{\sigma} & z_{\sigma} \\
  x_{t} & z_{t}
\end{vmatrix}
\]

is nonvanishing, one can replace the parametric variables \(\sigma\) and \(r\) by the coordinates \(x\) and \(z\). A nonvanishing Jacobian can be obtained by simply choosing the function \(z(\sigma)\) to avoid this case. Also, since this can be assumed to hold for the reference solution, it will hold also for perturbations of this solution. A differentiable surface function \(y(x, z)\) is thereby obtained. Moreover, by the above result for the vanishing of \(U, V\), and the nonvanishing of the Jacobian, one has

\[ w = \frac{\partial y}{\partial x}, \quad v = \frac{\partial y}{\partial z}, \]

where this refers to the actual solution \(y\) rather than the reference solution.

Similarly, as long as

\[
\begin{vmatrix}
  x_{\sigma} & z_{\sigma}^r \\
  x_{t} & z_{t}^r
\end{vmatrix} \neq 0,
\]

the solution may be parameterized locally by \(z^r\) and \(x\). This implies that \(z(x, y)\) is a well defined function, i.e. that the surface is not self-occluding. Thus it has been established that there is a range of possible solutions specified (1) by the choice of \(z(x, y)\) on the limb, and (2) by the choice of a combination of the parameters \(s(x, y)\) and \(r(x, y)\) on this boundary. There is thus a two parameter ambiguity for the shape solution in the neighborhood of the limb, as opposed to the one parameter ambiguity around an interior image curve [13].

The evolution equations for \(P\) are nonsingular on the limb, and in fact there is no problem in extending them past the limb, as long as the parameters of the reference solution are known in this unseen region. For a characteristic strip that intersects the limb, \(s\) and \(r\) will be finite at the point of intersection since the right hand sides of their evolution equations are bounded. Since however \(y_z = 0\) there, \(w = y_x\) and \(v = y_z = 0\) are forced on the limb, as stated earlier.

It is also possible to generate arbitrary numbers of solutions by specifying the depth on an interior image curve that terminates at the limb. This fact is used for the example image of the next section.

8. SHAPE FROM SHADING AS A PARTIALLY ILL-POSED PROBLEM

It is demonstrated numerically in this section that image regions where shape from shading is ill-posed appear in images of very simple objects, and that this occurs even though the image contains the occluding boundary and singular points. Our example is essentially an elongated egg-shape, viewed from a direction reasonably distinct from the long axis of the egg. This numerical result agrees with intuition based on the results of the last section.

The surface we consider is, in a body-centered coordinate system,

\[
x_x^2 + y_y^2 + \frac{|z_0|^3}{27} = 1.
\]

In this coordinate system, the light source direction is

\[
\dot{L}_b = (1, -1, 5),
\]

and the viewing direction is

\[
\nu_b = (-\sin, 0, \cos), \quad \sin = .15, \quad \cos = (1 - \sin^2)^{1/2}.
\]

The object is considered to be at positive \(z\) with respect to the viewer. The flow of the characteristic strips projected into the image plane is shown in Fig. 3. The lines without
FIG. 3. The characteristic flow on an object for which shape from shading is partially ill-posed, as projected in the image plane.

arrows represent the flow in the region of the object that is visible but not illuminated. Here the \(x\) direction is represented along the vertical axis, the \(y\) direction on the horizontal axis, and the \(z\) direction is into the page.

The image of this object contains a small potentially ambiguous region of the type described in Section 6, and illustrated in Fig. 1. This region is bounded on one side by a characteristic strip line, and on the other by the limb. The bounding characteristic strip segment touches the limb at both ends. It is a part of a strip which originates at the singular point, touches the limb, and reenters the image to finally exit elsewhere. The bounded image region is isolated: it is unconnected to the rest of the image by any characteristic strip in the image. Thus the shape in this region is potentially ambiguous. From now on we focus exclusively on this region, which we refer to as the ill-posed region of the image.

Below, we use the viewer coordinate system in which the \(z\) axis is aligned with the viewing direction. This coordinate system is related by a rotation to the body-centered system. The ill-posed region of the imaged object is displayed in Fig. 4, projected in the \(x-z\) plane of the viewer system. The \(x\) axis is again vertical. The projection is done in this plane rather than the image plane, because otherwise the singularity of the projection near the occluding boundary would make the figure difficult to interpret. The flow of steepest ascent curves on the object, corresponding to the characteristic strips, is also indicated. The occluding boundary is represented by the lower curved line; the visible region of the object is above this line. The characteristic strip bounding the ill-posed region is the upper straight line, and was obtained by numerical integration. It becomes tangent to the occlud-

FIG. 4. The characteristic flow in the ill-posed region of the object of Fig. 3, projected into the \(z-x\) plane.
surface solution to the limb. We have computed the characteristic strips using the evolution equations both in their singular version of Eq. (9), and in the nonsingular version of Eqs. (13) and (26), obtaining consistent results. For the exact solution, which we computed as a check, the accuracy of the numerical integration is better than $10^{-12}$.

The results of integrating the characteristic strips starting from the initial line are shown in Fig. 5 for the exact solution. These are just the integral curves of the flow shown in Fig. 4. Note that the characteristic strips in the ill-posed region are indistinguishable from straight lines when projected in the $x$--$z$ plane.

Next, $z$ was perturbed on the initial line by

$$z(y) = z_s + f(y - y_c)^2,$$

where $z_s$ is the standard value of $z$ obtained by solving Eq. (38) for the given values of $x$ and $y$, $f$ is an arbitrary constant giving the scale of the perturbation, and $y_c$ indicates the $y$ coordinate of the point where the initial line intersects the bounding base characteristic.

The perturbation was chosen quadratic so that both it and its derivative with respect to $y$ vanish at $y = y_c$, on the bounding characteristic strip. This ensures that the solution in the ill-posed region will join smoothly onto the fixed solution in the rest of the image. Also, at the limb, the perturbation has finite derivative with respect to $y$. Therefore, $y_z = 0$ and $y_y$ is the same as for the standard solution, as is required since this quantity is fixed by the limb.

In Fig. 6, we display our results for a perturbation scaled by $f = 10^6$. With $f$ of this magnitude, the perturbation is comparable in size to the amount by which $z$ varies over the initial line in the standard solution. Again, the leftmost line is the bounding characteristic strip, and the horizontal line is the initial line. The occluding boundary is not displayed explicitly since its depth is not known a priori, but it is given by the smooth curve passing through the ends of the displayed strips. In Fig. 7, the characteristic strips are displayed in terms of the coordinates $x$ and $z$. The fact that they do not intersect shows that the recovered surface is not self-occluding from the viewer's perspective. Also, the length of the characteristic strips is approximately proportional to the evolution time, so the initial line under time evolution remains transverse to the characteristic strips, and the Jacobians of Eqs. (35) and (37) remain nonzero as required. Finally, in Figs. 8--11 the recovered surface is shown from various angles. The perturbed surface recovered for the ill-posed region is shown by itself, and together with the known solution of Eq. (38). The perturbed surface joins smoothly onto the standard solution along one side, as it must since they both share the bounding characteristic. However, the difference in the two surfaces is clear at the limb, where there is a sharp discontinuity from the perturbed to the standard surface. The scale of the display has been exaggerated for better visibility.
FIG. 7. The characteristic strip lines for the object of Figure 3, perturbed with $f = 10^6$, projected into the $z-x$ plane.

FIG. 8. A rear view of the surface recovered for $f = 10^6$, corresponding to the ill-posed region of the object in Fig. 3. The position of the viewer is into the page.

FIG. 9. The same view as in Fig. 8 of the perturbed surface, shown together with the unperturbed surface. Note the sharp discontinuity at the occluding boundary.

FIG. 10. A front view of the surface recovered for $f = 10^6$, corresponding to the ill-posed region of the object in Fig. 3. The position of the viewer is out of the page.

FIG. 11. The same view as in Fig. 10 of the perturbed surface, shown together with the unperturbed surface. Note the smooth joining of the two surfaces at a line corresponding to the bounding characteristic strip of the ill-posed region. The slight roughness of the join is due to the fact that the perturbed surface has been averaged over nearest neighbors to fill in holes.

In Figs. 12–17 are shown the equivalent results for a perturbation scaled by $f = 2(10^6)$. Clearly, this yields yet another surface solution in the ill-posed region. These new solutions were derived for arbitrary perturbations of the depths on the initial image line described above. Thus, it is almost certain that similar results would have been obtained for many different choices of initial conditions, and that shape from shading is in fact ill-posed for this region.

The ill-posed region described above was a very small fraction of the whole image. We now present a second image example which demonstrates that ill-posed regions need not always be tiny. However, it is somewhat contrived, since the sole singular point in the image is very close to the limb, and such an image would anyway be very difficult to reconstruct because of the large gradients.

In a body-centered coordinate system, the object is given by

$$A + B^2 + \frac{z^2}{64} = 1,$$
FIG. 12. The characteristic strip lines for the object of Fig. 3, perturbed with \( f = 2\times 10^9 \), projected into the \( z-x \) plane.

FIG. 13. The characteristic strip lines for the object of Fig. 3, perturbed with \( f = 2\times 10^9 \), projected into the \( z-x \) plane.

FIG. 14. A front view of the surface recovered for \( f = 2\times 10^9 \), corresponding to the ill-posed region of the object in Fig. 3. The position of the viewer is out of the page.

where

\[
A = \begin{cases} 
0.8x + 0.56 & x \geq -0.5 \\
(x + 0.9)^2 & x < -0.5 
\end{cases},
\]

\[
B = \begin{cases} 
y & x \geq 0 \\
y + x^2 & x < 0 
\end{cases}.
\]

The illumination is from above, with \( \hat{L} = -\hat{z} \). The characteristic strips are therefore the lines of steepest descent, or fall lines, in \( z \). The viewing direction is

\[
(0, \cos \theta, -\sin \theta),
\]

with \( \sin \theta = 0.15 \). There is a unique singular point in the image, which is a source.

The illuminated portion of the object is displayed projected into the \( x-y \) plane in Fig. 18, together with the flow field for the characteristic strips. In this figure, the standard convention (\( x \) axis horizontal, \( y \) axis vertical) is used, unlike the previous figures. The visible points are identified by complete arrows, while those points that are occluded in the image have partial arrows. The occluding boundary can be located roughly as a line dividing the

FIG. 15. The same view as in Fig. 14 of the perturbed surface, shown together with the unperturbed surface. Note the smooth joining of the two surfaces at a line corresponding to the bounding characteristic strip of the ill-posed region.
regions of complete and incomplete arrows. The characteristic strips are the integral curves of this flow. The singular point is at the lower left, between the opposed arrows.

Consider the characteristic strips in the upper right portion of Fig. 18. It should be fairly clear that they cross the occluding boundary at the left heading rightwards into the visible region, and exit the image at the right boundary (the shadow boundary). Thus the whole upper visible region in Fig. 18 is unconnected to any singular point in the image, and is an ill-posed region of the type discussed above.

The flow of characteristic strips in the image plane is displayed in Fig. 19. The sole source is located at the top of the figure. Again, the behavior of the characteristic strip lines should be readily apparent. The horizontal $x$-axis is the same in both figures, so that the strips in the figures can be identified according to the $x$-coordinate at which they exit the image. It is clear that the ill-posed region in the image is not as dramatic as it seemed in Fig. 18.

FIG. 16. A rear view of the surface recovered for $f = 2(10^9)$, corresponding to the ill-posed region of the object in Fig. 3. The position of the viewer is into the page.

FIG. 17. The same view as in Fig. 16 of the perturbed surface, shown together with the unperturbed surface.

FIG. 18. A second example of an object with an unconstrained image region, viewed from above and projected into the $x$-$y$ plane.

18. It is probably on the order of a fifth of the whole image, bounded on the left by a characteristic strip, and on the right by the image boundary. We have not actually verified that there are multiple solutions to shape from shading for the identified ill-posed region in this example. The example is intended to demonstrate only that such regions can be respectable fractions of the whole image. When the image described

FIG. 19. The same object as in Fig. 18, projected into the image plane.
Lemma 1. Every smooth, closed, orientable surface \( S \) embedded in \( R^3 \) can be approximated arbitrarily well by one whose height function is a Morse function [14, 15].

Proof. Let \( h \) denote the height function on the surface \( S \). We use the following result. Theorem. Every smooth, real-valued function on a compact manifold \( M \) can be uniformly approximated by a Morse function on \( M \) (see, e.g., [14, 15]). Let \( g = h + \delta h \) be such a Morse function approximation to \( h \). We may take

\[
\delta h = a \cdot (x, y, z),
\]  

(45)

for some vector \( a \) of arbitrarily small magnitude [14]. Define a \( C^\infty \) function \( e(x) \) satisfying \( |e(x)| \leq 1 \) and

\[
e(x) = \begin{cases} 
0 & |x| \leq \cos(\theta_2) \\
1 & |x| \geq \cos(\theta_1) 
\end{cases},
\]

where \( 0 < \theta_1 < \theta_2 < \pi/2 \). Denote by \( \cos(\theta) \) the real-valued function on the surface \( S \) which gives the cosine of the angle between the height direction \( \hat{z} \) and the surface normal \( \hat{n} \) for each point on \( S \). Thus, \( \cos(\theta) = \hat{n} \cdot \hat{z} \).

Finally, define the function

\[
g' = h + e(\hat{n} \cdot \hat{z}) \delta h.
\]  

(46)

We claim that \( g' \) is a Morse function on \( S \) if \( |a| \) is chosen sufficiently small. This is fairly clear since \( g' = g \) in the region where there are no critical points, and \( g = h \) only in the region with \( \hat{n} \cdot \hat{z} \leq \cos(\theta_2) \), where \( h \) has no critical points.

What we need to show is that \( g' \) also has no critical points in the region \( R \), where \( |\hat{n} \cdot \hat{z}| \leq \cos(\theta_1) \). At any point \( p \) in this region, one can erect a local coordinate system \((\xi_1, \xi_2)\), given by the projection of the surface onto the directions

\[
\hat{\xi}_1 = \frac{\hat{n} \times (\hat{z} \times \hat{n})}{|\hat{n} \times (\hat{z} \times \hat{n})|},
\]

\[
\hat{\xi}_2 = \frac{\hat{z} \times \hat{n}}{|\hat{z} \times \hat{n}|}.
\]

(47)

\( \hat{\xi}_1 \) lies in the direction of steepest ascent. To demonstrate that \( p \) is not a critical point of \( g' \), consider the derivative

\[
\frac{\partial g'}{\partial \xi_1} |_{\xi_2} = \frac{\partial h}{\partial \xi_1} + a \cdot \frac{\partial r}{\partial \xi_1} \varepsilon + a \cdot r \frac{\partial e}{\partial \xi_1},
\]

(48)

where

\[
r = (x, y, z).
\]

(49)

We have

\[
\frac{\partial r}{\partial \xi_1} |_{\xi_2} = \hat{\xi}_1,
\]

(50)

and therefore

\[
\frac{\partial h}{\partial \xi_1} |_{\xi_2} = \hat{\xi}_2 - |\sin(\theta)| \geq \sin(\theta_1),
\]

since \( h = z \). One can find a bound \( B \), such that

\[
\left| \frac{\partial e}{\partial \xi_1} \right| < B,
\]

(51)

and a bound \( B' \) such that \( |r| < B' \), for every point in the region \( R \), since it is compact. Then if

\[
|a| < \frac{\sin(\theta_1)}{4BB'},
\]

it is clear that

\[
\frac{\partial g'}{\partial \xi_1} |_{\xi_2} \neq 0,
\]

(52)

at every point in \( R \), so that it contains no critical points. Therefore \( g' \) is a Morse function on \( S \).

Lastly, we find a surface which has \( g' \) as its height function. The existence of such a surface proves the lemma. Consider the regions of the surface where

\[
|\cos(\theta)| > \cos(\theta_3),
\]

(53)

where \( 0 < \cos(\theta_3) < \cos(\theta_2) \). For points in these regions, one can erect local coordinate systems parameterizing the surface by its projection in the \( x-y \) plane. Define the perturbed surface to be given by the graph of \( g' \) over the region above, and equal to the original surface over the complement of this region. The join of these two surfaces is smooth, since \( g' = h \) in a neighborhood of the boundary between them.

Q.E.D.

Also, note that if the original surface \( S \) in the above lemma is non-self-intersecting, then one can choose the surface perturbation small enough so that the approximating surface is also non-self-intersecting.

Lemma 2. Let \( S \) be a smooth, closed surface in \( R^3 \), whose height function is a Morse function. Then \( S \) can be approximated arbitrarily well by a surface whose height function is a Morse function, and on which there are no saddle connections.
\textbf{Proof.} Suppose that $S$ has a saddle connection line $l$. By definition, $l$ is the intersection of the stable manifold of one saddle point with the unstable manifold of another, for the dynamical system

$$\frac{dr}{dt} = \dot{z} - (\hat{n} \cdot \dot{\xi}) \hat{n}. \quad (54)$$

We will demonstrate that a small perturbation in the surface can shift the unstable manifold line slightly, so that it no longer intersects the stable manifold line from the upper saddle point.

Choose a point $p$ on $l$ such that the surface normal $\hat{n}$ at $p$ obeys $1 > |n_z| > 0$. Such a point $p$ must exist since the saddle points are nondegenerate. In some neighborhood of $p$, the surface can be parameterized by coordinates $\xi_1$, $\xi_2$ such that (1) the flow lines in these coordinates are straight lines in the $\xi_1$ direction, and (2) $\partial z / \partial \xi_1 = 1$ along these lines, so that the lines of constant $\xi_1$ are also lines of constant $z$, (3) $p$ is the origin. This follows from the Tubular Flow Theorem [6]. The $\xi$ coordinates are closely related to those in the previous lemma. $l$ over this neighborhood is given by $\xi_2 = 0$.

Define a rectangular region $R$ in the $\xi$ coordinates small enough so that the corresponding surface region can also be parameterized by $x$ and $y$. Then, the $\xi$ coordinates can be thought of as parameterizing a region of the $x$-$y$ plane. Label this region of the $x$-$y$ plane by $R$. We will now define a smaller region $R' \subset R$ in the $x$-$y$ plane, on which the surface will be perturbed. In the $\xi$ coordinate system: (1) $R'$ is rectangular, (2) $R$ and $R'$ are positioned symmetrically about the origin $p$, (3) $(L_1, L_2)$ and $(L_1', L_2')$ respectively represent the dimensions of $R$ and $R'$.

Within $R'$, $z(\xi_1, \xi_2)$ will be perturbed by $\delta z(\xi_1, \xi_2) = \psi(\xi_1) \phi(\xi_2)$, $\psi$ and $\phi$ are non-negative $C^\infty$ functions with the following properties:

\begin{align*}
\phi(\xi_2) = 0, & \quad |\xi_2| \geq L_2' / 4,
\frac{\partial \phi}{\partial \xi_2} > 0, & \quad |\xi_2| \leq L_2' / 8,
\psi(\xi_1) = 0, & \quad |\xi_1| \geq L_1 / 4,
\psi(\xi_1) \neq 0, & \quad |\xi_1| < L_1 / 4.
\end{align*}

Also, we assume $|\partial \psi / \partial \xi_1|$ is bounded by $H$, and $|\partial \phi / \partial \xi_2|$ by $F$.

The surface strip entering the region $R'$ at $(-L_1 / 2, 0)$ originates at a saddle point, by assumption, and is part of the unstable manifold of this point. On the perturbed surface, we refer to this line and its continuation as $l'$. We are interested in finding the $\xi_2$ value at which $l'$ exits the region $R$. First, an equation for the flow in the $\xi$ coordinates is derived.

$$\frac{d}{dt} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \frac{\partial (\xi_1, \xi_2)}{\partial (x, y)} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}, \quad (56)$$

where

$$\frac{\partial (\xi_1, \xi_2)}{\partial (x, y)} = \begin{pmatrix} \frac{\partial \xi_1}{\partial x} & \frac{\partial \xi_1}{\partial y} \\ \frac{\partial \xi_2}{\partial x} & \frac{\partial \xi_2}{\partial y} \end{pmatrix}. \quad (57)$$

On the perturbed surface, from Eq. (54),

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = N \begin{pmatrix} \frac{\partial (z + \delta z)}{\partial x} \\ \frac{\partial (z + \delta z)}{\partial y} \end{pmatrix}, \quad (58)$$

with $N$ a normalizing factor,

$$N = \left[ 1 + \left( \frac{\partial (z + \delta z)}{\partial x} \right)^2 + \left( \frac{\partial (z + \delta z)}{\partial y} \right)^2 \right]^{-1}. \quad (59)$$

Finally, one obtains

$$\frac{d}{dt} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = N \begin{pmatrix} \frac{\partial (\xi_1, \xi_2)}{\partial (x, y)} \end{pmatrix}^T \begin{pmatrix} \frac{\partial (z + \delta z)}{\partial \xi_1} \\ \frac{\partial (z + \delta z)}{\partial \xi_2} \end{pmatrix}. \quad (60)$$

To evaluate the matrix appearing in this equation, we use the fact that

$$\frac{\partial (\xi_1, \xi_2)}{\partial (x, y)} = \begin{pmatrix} \frac{\partial \xi_1}{\partial x} & \frac{\partial \xi_1}{\partial y} \\ \frac{\partial \xi_2}{\partial x} & \frac{\partial \xi_2}{\partial y} \end{pmatrix}^{-1}. \quad (61)$$

The $\dot{\xi}_1$ direction in the $x$-$y$ plane is given by $\nabla z$, since by assumption this coordinate parameterizes lines following the gradient of $z$. The $\dot{\xi}_2$ direction is perpendicular to this, since $\xi_2$ parameterizes lines of constant $z$. Therefore, $\dot{\xi}_1$ and $\dot{\xi}_2$ are given by Eq. (47), and the two rows of $\partial (\xi_1, \xi_2) / \partial (x, y)$ are orthogonal. This implies
for some other small quantity \( \delta \). The time during which \( l' \) remains in \( R' \) is less than

\[
t_{\text{max}} = \frac{L_1'}{A(1 - \delta)}. \tag{69}
\]

Last, we also require that

\[
(FH) \frac{(L_2')^2}{A(1 - \delta)} < \frac{L_2'}{8}. \tag{70}
\]

Then the trajectory \( l' \) always satisfies

\[
\xi_2 < \frac{L_2'}{8}, \tag{71}
\]

while it passes through \( R \). Since \( \frac{\partial \xi_2}{\partial t} \) is nonnegative in this region, \( l' \) is displaced exclusively towards the positive \( \xi_2 \) direction. Thus, \( l' \) exits \( R \) at \( (L_1/2, \xi_2) \) with \( \xi_2 > 0 \).

On the other hand, the trajectory originating as the stable manifold of the upper saddle point passes through the point \( (L_1/2, 0) \). The later history of \( l' \) extends to higher \( z \), and therefore cannot include this point. Thus, the stable and unstable manifolds of the two saddles points no longer intersect. It is also clear that, for a small enough surface perturbation, no critical point has been introduced. Repeating this argument for every saddle connection of the original surface, one finally obtains a surface with no saddle connections.

Q.E.D.

Last, to finish the proof of the theorem of Section 5, we must show that a perturbed structurally stable surface remains structurally stable, if the perturbation is sufficiently small. Consider therefore a structurally stable surface \( S \), and let \( S' \) be a small perturbation of this surface. This means that \( S \) and \( S' \) are diffeomorphic, with a diffeomorphism close to the identity. Consider the vector field \( V_S \) given by \( \xi - (\xi \cdot \hat{z}) \hat{n} \) evaluated on \( S \), and also the vector field \( V_{S'} \) given by evaluating this expression on \( S' \). Let \( V_{S'} \) be the pull-back of \( V_S \) onto the surface \( S \). \( V_{S'} \) constitutes a perturbation of the vector field \( V_S \) on \( S \). By assumption, \( V_S \) is a structurally stable vector field on \( S \). But the theorems of Peixoto, and of Palis and Smale [6, 8], state that a sufficiently small perturbation of a structurally stable vector field is also structurally stable. Thus, \( V_{S'} \) can be assumed structurally stable. But this implies that \( S' \) is structurally stable. This proves the theorem.

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REFERENCES


