Handout—Note on converting binary periodic fractional values to decimal

Consider a periodic binary fractional value, like \((10.00011)_2 = 2.1\). Here is a simple derivation regarding how to convert it into a rational number, written as a decimal fraction (e.g., \(21/10\)).

First, assume that our binary periodic number is within 0 and 1, and that it has got no anti-period—that is, the periodic part comes immediately to the right of the binary point, as in \((0.0011)_2\).

If \(k\) is the length of the period, the general form for numbers of this sort is \(x = (0.b_1b_2\ldots b_k)_2\). Notice that multiplying \(x\) by \(2^k\) will bring a copy of the period to the left of the binary point, while at the same time leaving the period unchanged to the right of point, i.e.

\[
x \cdot 2^k = (b_1b_2\ldots b_k, b_1b_2\ldots b_k)_2
\]

For example, multiplying \((0.0011)_2\) by \(2^4 = 16\) gives \((11.0011)_2\).

Now, note that \(x \cdot (2^k - 1)\) is an integer, since subtracting \(x\) from \(x \cdot 2^k\), the period cancels out:

\[
\begin{align*}
b_1\ldots b_k & \cdot \overline{b_1\ldots b_k} \\
- & 0 \cdot \overline{b_1\ldots b_k} \\
= & b_1\ldots b_k
\end{align*}
\]

Letting \(v\) be the integer whose binary representation is \((b_1\ldots b_k)_2\), we get \(x \cdot (2^k - 1) = v\), from which \(x = v/(2^k - 1)\). Thus, for example, \((0.0011)_2 = 3/15 = 1/5\).

Next, consider the case of a binary periodic number within 0 and 1 with length-\(l\) anti-period, i.e., a number of the form \(y = (0.c_1\ldots c_l b_1b_2\ldots b_k)_2\). Letting \(u\) be integer whose binary representation is \((c_1\ldots c_l)_2\), we can reduce back to the first case by multiplying \(y\) by \(2^l\) and subtracting \(u\) from it, since \(x = y \cdot 2^l - u\) is clearly a binary periodic number within 0 and 1 with no anti-period:

\[
\begin{align*}
c_1\ldots c_l & \cdot \overline{b_1\ldots b_k} \\
- & c_1\ldots c_l \cdot \overline{b_1\ldots b_k} \\
= & 0 \cdot \overline{b_1\ldots b_k}
\end{align*}
\]

Once the decimal fraction corresponding to \(x = (0.b_1b_2\ldots b_k)_2\) as been computed, we can recover \(y\) as \((x + u)/2^l\). For example, \((0.00011)_2\) has anti-period of length 1, so we can write \(x = 2y\) (since \(u = 0\) in the example), and from the fact that \((0.0011)_2 = 1/5\) (as computed above), we get \(y = x/2 = 1/10\).

Last, we get to the most general case, i.e., numbers of the form \(z = (d_{m-1}\ldots d_0, c_1\ldots c_l b_1b_2\ldots b_k)_2\). Letting \(t\) be the integer whose binary representation is \((d_{m-1}\ldots d_0)_2\), we easily reduce to the second case above by looking at \(y = z - t = (0.c_1\ldots c_l b_1b_2\ldots b_k)_2\):

\[
\begin{align*}
d_{m-1}\ldots d_0 & \cdot \overline{c_1\ldots c_l b_1\ldots b_k} \\
- & d_{m-1}\ldots d_0 \cdot \overline{c_1\ldots c_l b_1\ldots b_k} \\
= & 0 \cdot \overline{c_1\ldots c_l b_1\ldots b_k}
\end{align*}
\]
After figuring out the decimal fraction corresponding to \( y \), we can recover \( z \) as \( t + y \). For example, 
\[(10.000\overline{1})_2\] has integer part \( t = (10)_2 = 2 \), and so its representation as a decimal fraction is \( 2 + 1/10 = 21/10 \).

At this point we can add two periodic binary fractional values by converting them into fractions, and adding the results. In some cases, the (less formal) approach of doing binary addition on the infinite pattern might still work: try figuring out the result of adding \((0.10\overline{1})_2 + 0.0\overline{1}\overline{0}_2\) and make sure you understand what happens.